Fundamentals of Cost and Risk that Matter to Pension Savers and Life Annuitants

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Abstract

This chapter analyzes two types of investment strategies for an investor with a savings plan for retirement. In one, the investor sets an upper target which his wealth should not go above. In the other, the investor adds a lower target which his wealth should not go below. The analysis is done in a Black-Scholes model with one risky stock and one risk-free bond, with the restriction that the investor cannot invest more than his current wealth in the risky stock. We illustrate the results by describing a 30-year horizon. We use quantiles of the terminal wealth distribution or the level of accumulated wealth obtained by a given percentage of investors who follow the recommended strategy. The embedded guarantee and freedom to choose the upper and lower bounds represent the main appeal of this approach. We discuss the connection between expected return, affordable risk, and transparent fees for funds management.

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This chapter seeks to increase a pension saver's certainty about the amount of wealth available at retirement. To achieve this, we show how to design new, transparent investment strategies to be followed prior to the anticipated retirement date. Since reducing uncertainty about the amount of retirement wealth has a financial cost, we analyze this cost/risk tradeoff.

To determine one such investment strategy, the pension saver begins by choosing an upper bound for his retirement wealth, which is constrained to be no higher than this bound. The surprising consequence of this approach is that wealth is more likely to attain this bound than if there were no such bound. For example, suppose an investor chooses an upper bound of $100,000, and following the resultant optimal investment strategy means that the probability of retiring with exactly $100,000 is 70 percent. Following this strategy means that the investor will never end up with more than $100,000 at retirement. On the other hand, throwing away the constraint of the upper bound gives a different optimal investment strategy that has a probability of 50 percent of producing a retirement wealth of $100,000 or higher.

Our proposed investment strategies are fully described. They can be implemented through a customized investment product, such as a managed account or a target date fund, or by a sophisticated individual. They may enable pension savers to plan better for their retirement, since there is more certainty about how much money will be available at retirement.

We consider a pension saver who makes a one-off contribution and intends to retire in 30 years' time. (The results can be easily extended to allow for additional contributions.) The single contribution is invested in line with one of the investment strategies that incorporate either an upper bound and/or a lower bound on the wealth at the retirement date. The distribution of retirement wealth is analyzed.

Background and Motivation

At least four factors threaten the understanding of pension savers and annuitants:

(1) Duration: there is always uncertainty about the retirement date and for how long the individual will live.

(2) Return with risk-adjusted measures: while there is no widespread agreement on how best to measure investment performance, we believe both risk and return should be measured as quantiles rather than, for example, as expected values.

(3) Volatility: volatile investment returns imply uncertainty about the accumulated amount of wealth at retirement, which is anathema to some savers. Furthermore, many pension savers are averse to losing money.

(4) Fees: the main problems of investment management costs are a lack of pricing transparency and the pension saver's ignorance about the amount of investment returns lost due to portfolio administration fees.

Our approach to the study of retirement savings relies on three concepts. First, any comparison of investment strategies must adjust for performance risk. By performance risk, we mean the possibility of large losses. Most studies on the performance of savings plans lack such a risk-adjustment component (Gerrard et al. 2014; Guillén et al. 2013, 2014; De Franco and Tankov 2011; Benartzi and Thaler 1999). Second, the management of longevity risk is usually done by a risk transfer to a third party at a very large cost. We claim that pooling longevity risk can potentially produce significant cost savings to the pensioner (Donnelly et al. 2014). Third, management fees have an important effect on wealth at retirement. Many authors show that the fees paid for managing investment funds and for mitigating longevity risk produce a substantial decrease to the wealth of pensioners (Guillén et al. 2014; MacKay et al. 2015).

A sensible investment strategy for a pension saver to retirement and beyond requires analyzing the stochastic distribution of retirement wealth (Basu et al. 2011; Greninger et al. 2000; Browne 1999; Grossman and Zhou 1996). The accumulated wealth during savings and consumption at a given point of time depends on individual decisions and market conditions that investment assets have been through over time. A consumer living during a decade of economic expansion can have a different retirement wealth compared to another pensioner who may have lost most of his assets living through bubbles and recessions. Their lifetime investment experiences correspond to two trajectories with opposite consequences (Jin and Zhou 2008; Bodie et al. 1992).

We consider the problem of setting a dynamic investment strategy to invest initial wealth or, alternatively, periodic amounts, in order to reach a target capital sum at the intended retirement date. Many studies have introduced constraints on the portfolio or the terminal wealth (Bouchard et al. 2010; Van Weert et al. 2010; Gaibh et al. 2009; Boyle and Tian 2007; Cuoco 1997; Korn and Trautmann 1995; Zariphopoulou 1994). Ours is a different formulation of one of the problems described in Dhaene et al. (2005), in which an investor wishes to find the optimal constant-proportion portfolio that attains the highest target capital with a fixed probability. We introduce a different vision of risk, since we consider both upper and lower targets. We constrain the investor to have *at most* a target capital at the time of retirement, whereas Dhaene et al. (2005) focuses on ensuring that at least the target capital is attained with maximum probability. We also propose a more general setting in which we have both an upper and a lower target, so that the pensioner can ensure that the wealth at retirement lies between these bounds.

In proposing this mechanism, we seek to promote transparent and automatic products where pension savers can obtain guaranteed bounds on their potential final wealth, and where their portfolios are automatically rebalanced to the chosen strategy. As noted by Leshno and Levy (2002) and von Gaudecker et al. (2011), there is much heterogeneity regarding risky choice behavior in the population. Thus, if decision making is made automatic, this reduces uncertainty and cost. Our ultimate aim is to combine the proposed strategies in the savings phase with financial management after retirement, in order to design annuity schemes where the longevity risk is also inexpensive. For instance, in some annuity schemes, the risk is shared among fund participants instead of being transferred to the insurance market (Donnelly et al. 2013, 2014).

Innovations in the design of pension products are rare in Europe, where there is generally a subsistence (or higher) level of income provided by public or occupational pensions. The private pension sector uses mostly classical products, namely, pure financial investments before retirement with some tax saving rewards, and, after retirement, the purchase of an annuity, with not much more to be offered to customers.

The importance of investment performance for lifetime investors is widely recognized (Milevsky and Huang 2011). Pension savers invest their savings for long periods of time, often for several decades. Small deviations in performance are magnified hugely as returns are compounded over decades. Current performance evaluation methods are usually too myopic and they overlook persistence in performance, for example, the role of time that we also discuss here.

Foundations of Our Model and the Proposed Strategy

Our analysis sets limits on the level of wealth at the retirement date; we call this the restricted case. These limits constrain the stochastic distribution of retirement wealth and result in a risk-return profile that is significantly different from the unrestricted situation. We set limits on the level of wealth at the retirement date and compare the restricted and unrestricted situations.

Figures 9.1 and 9.2 compare those two scenarios. Figure 9.1 presents the unrestricted case, showing two pension savers over 30 years with no restrictions on their retirement wealth. One saver is very unlucky to have much less than the initial investment, as shown by the grey line, and another is fortunate and ends up with a very successful investment, as shown by the solid black line. For comparison with the restricted case, the bounds for the latter are also shown in Figure 9.1. The bounds happen to be each crossed by the unrestricted simulations.

Insert Figures 9.1 and 9.2 here

Figure 9.2 shows an upper and a lower bound on retirement wealth, indicated by the horizontal dashed and dotted lines, for the restricted case. This means that the trajectories of the accumulated wealth are constrained to stay between the bounds. In particular, wealth cannot fall below the lower bound at retirement: this is the case for the light grey path. This means that whenever the discounted value of the lower bound is reached, the investor chooses to invest only in bonds (in these simulations, the annual discount factor equals one and so the bounds are constant across the 30 years). The second path, shown by the dark grey line, is the situation where the discounted value of the upper bound is crossed before retirement. It is very similar to the case with a lower bound: once the discounted value of the upper bound is reached, all wealth is invested in the risk-free bond. In the third case, shown by the solid black line, the investor has both an upper and a lower bound. Accumulated wealth must not fall below the lower level, or exceed the upper level. While, potentially, better gains could have been obtained, it is guaranteed that the investor is accumulating at least a minimum value and at most a maximum value.

Technical details of the calibrating strategy are given in the Appendix, and more detail appears in Donnelly et al. (2014, 2015a, and 2015b).

Numerical Illustration

Here we investigate the optimal strategy for the constrained strategy. We use the unrestricted strategy as the benchmark strategy. Our initial investment is 300 units[1] and the time horizon is 30 years. Table 9.1 shows the distribution of the terminal wealth under the restricted strategy for combinations of the lower and upper bounds. The probabilities are expressed by the first column on the left as a percent. The subsequent columns refer to the corresponding percentiles under each of the scenarios.

Insert Table 9.1 here

The unrestricted case appears in the second column. Here, one investor has 300 units at the beginning and after 30 years has a 1 percent chance of having at most 82.09 units. There is a 40 percent chance of having at most 473.87 units. Following the unrestricted strategy means that there is just over a 20 percent chance of ending up with less than the initial investment. There is slightly less than a 50 percent chance of at least doubling the initial investment after 30 years. Note that we do not allow for inflation and the risk-free interest rate is zero. We assume only that the risky stock has a larger expected return than a risk-free bond, and we have computed these examples with realistic scenarios of volatility.

The third column in Table 9.1 shows the case where we set only a lower bound of 250. That means that the investor loses at most 50 units from the original investment of 300, by the end of the 30 years, which happens 30 percent of the time. Imposing the lower bound means that the chance of at least doubling the initial investment falls to just under 30 percent. In other words, the cost of having a guaranteed floor is the loss of potential high gains.

In the fourth column, we set only an upper bound of 587.10, which is close to being double the original investment; the upper bound is attained 50 percent of the time. The effect of imposing the upper bound is to raise the lower quantiles, in comparison to the unrestricted case. However there is a 1 percent chance that the investor ends up having accumulated only 100.86 units, which means that he would have lost two-thirds of the initial investment.

The last column in Table 9.1 is our fully restricted case, with the two bounds. The main result about this case is that in 20 percent of the future scenarios, pension savers get exactly the lower bound of 250 units. In comparison, the unrestricted case has a similar percentage of scenarios in which the investor ends up with less than 250 units. On the other hand, in the fully restricted case only 30 percent of the investors approximately double the initial investment, and they never get above the ceiling of 587.10. In fact, 50 percent of the investors stay within those two bounds.

Let us use $F_{(L,U)}(\cdot)$ ·to represent the statistical distribution function of the terminal wealth for a given lower bound *L* and upper bound *U*. The interpretation is that the terminal wealth is at most equal to the value *x* with probability $F_{(L,U)}(x)$. Note that as the unrestricted case is equivalent to the restricted case with the bounds L=0 and U=∞, its distribution function is $F_{(0,\infty)}(\cdot)$. As before, we consider the unrestricted case as the benchmark.

Figure 9.3 shows three probability plots which allow us to compare visually the terminal wealth distributions of the restricted and unrestricted cases. In the left-most plot, the solid black line is a plot of $F_{(0,\infty)}(x)$ against $F_{(250,\infty)}(x)$—corresponding to a lower bound of 250 and no upper bound—as *x* varies. Each point on the line corresponds to a possible terminal wealth value, although the value is not explicitly shown on the plot. Instead, the *x*- and *y*-coordinates give the probability of the terminal wealth being at most that value under the restricted and unrestricted case, respectively. In all diagrams, the dashed line corresponds to a plot of $F_{(0,\infty)}(x)$ against itself, which we consider as the benchmark line.

Insert Figure 9.3 here

The solid black line beneath the dashed line indicates that the quantiles of the restricted case are below those of the unrestricted, and vice versa. For example, part of the horizontal segment of the solid black line is above the dashed line, indicating that the very lowest quantiles of the restricted case are above those of the unrestricted case. This corresponds to an investor's terminal wealth hitting the lower bound in the restricted case.

In the middle plot in Figure 9.3, the solid black line is a plot of $F_{(0,\infty)}(x)$ against $F_{(0,587.10)}(x)$ —corresponding to no lower bound and an upper bound of 587.10—as *x* varies. Here the lower quantiles for the restricted case are above those for the unrestricted case, and it is only when the upper bound bites that they start falling below. In the right-most plot, the restricted case has a lower bound of 250 and an upper bound of 587. This gives quantiles for the restricted case that generally lay below those of the unrestricted case and are in a shorter range. Figure 9.3 illustrates the ability of our proposed method to restrict the distribution of the terminal wealth to a limited domain.

Conclusion

The puzzle of retirement plan investment decisions in savings or in consumption phases is largely due to a weak comprehension of risk. We believe that it is much easier to communicate the extent of uncertainty using bounds on the desired income. By letting pension savers fix an upper and a lower wealth bound, we can produce an automatic method that fixes the amount of current wealth that should be invested in risky stock. This method produces an embedded guarantee that accumulated wealth is never smaller than the lower bound fixed by the pension saver, and it is never larger than the upper bound fixed by the pension saver. The investment strategy can be implemented by individuals or by fund managers.

Comparing strategies based only on their expected returns neglects the most dangerous part of the potential outcome, and we argue that decisionmakers should look at the extremes of the distribution instead. This is why we concentrate on quantiles, to set up target bounds so that the investment is restricted to remain within the limits and be able to compare distributions by means of a risk-adjusted judgment. The visualization of probability plots summarizes the power of investment strategies that can be implemented in practice.

When we study the effect of administration costs on the expected or the median rate of return as in Guillén et al. (2014), we fix the level of risk in the wealth distribution. The level of risk in the wealth distribution can be measured by a (low) percentile in the distribution (i.e., a Value-at-Risk measure). We then compare two distributions, one with no administration costs, and a second with positive administration costs, which has the same risk as the given baseline. Accordingly, we find the reduction in return due to administration costs exclusively such that the percentile of the wealth distribution with administration costs and without administration costs are the same. This method allows for risk-adjusted evaluations of the return lost due to the presence of managerial costs. Here we have seen that risk of terminal wealth in a pension saver setting can be bounded using suitable investment strategies. Therefore, our proposed mechanism does not need to implement and to pay for the fund manager to reduce risk.

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Appendix

We assume investment in a continuous-time market model over a finite time horizon $[0,T]$ for an integer $T>0$. We also refer to T as the *terminal time*. The market consists of one risky stock and one risk-free bond. The price of the stock is driven by a one-dimensional, standard Brownian motion $W=\{W(t);t\in[0,T]\}$. The risk-free bond has price process $\{S_0(t);t\in[0,T]\}$ and the risky stock has price process $\{S_1(t);t\in[0,T]\}$ with dynamics

$$\mathrm{d}S_0(t)=rS_0(t)\mathrm{d}t, \qquad \mathrm{d}S_1(t)=S_1(t)\big(\mu dt+\sigma dW(t)\big),$$

with $\sigma>0$, $S_0(0)=1$ and $S_1(0)$ being a fixed, strictly positive constant. We assume that $\mu>r$.

The information F_t available to investors at time *t* is generated by the Brownian motion up to time *t*. The market price of risk is $\theta \coloneqq (\mu-r)/\sigma$.

An investor starts with a fixed non-random initial wealth $x_0>0$ and plans to make a sequence of known future savings $a>0$. Define $C(t)$ to be the sum from time 0 to time t of the investor's planned discrete savings, with

$$dC(t)=\begin{cases} a & if \quad t=1,2,\ldots,T-1 \\ 0 & otherwise. \end{cases}$$

In other words, at the end of each unit time period, the investor pays an amount $a>0$ into their fund.

A portfolio process $\pi=\{\pi(t);t\in[0,T]\}$ is a square-integrable, $\{\mathsf{F}_t\}$ -progressively measurable process. The investor follows a self-financed strategy, investing at each instant $t\in[0,T]$ a monetary amount $\pi(t)$ in the stock such that $\pi=\{\pi(t);t\in[0,T]\}$ is a portfolio process.

The wealth process $X^{\pi}=\{X^{\pi}(t);t\in[0,T]\}$ corresponding to a portfolio process is the F_t-adapted process given by the *wealth equation*

$$dX^{\pi}(t)=\big(rX^{\pi}(t)+\pi(t)\sigma\theta\big)dt+\pi(t)\sigma dW(t)+dC(t), \quad X^{\pi}(0)=x_0.$$

Define the *savings plan* g of the investor, which is the discounted sum of the future savings by the investor by

$$g(t) := \int_t^T e^{-r(s-t)} dC(s), \quad \forall t \in [0,T].$$

Then the set of *admissible portfolios* for the investor's initial wealth $x_0 > 0$ is defined to be

$$\mathsf{A} := \{\pi : \Omega \times [0,T] \to \mathsf{R} : X^\pi(0) = x_0, \quad and \quad X^\pi(t) + g(t) \geq 0, \quad t \in (0,T]\}.$$

We say that a portfolio process π is *admissible* if $\pi \in \mathsf{A}$.

Define the state price density process H as $H(t) := \exp\left(-\left(r + \frac{1}{2}\theta^2\right)t - \theta W(t)\right)$, for each $t \in [0,T]$. A portfolio π must satisfy the *budget constraint* that

$$\mathsf{E}\left(H(T) X^\pi(T)\right) \leq x_0 + g(0).$$

The utility function of the investor is the power utility function

$$v(x) := \frac{1}{\gamma} x^\gamma, \qquad x > 0,$$

for a fixed constant $\gamma \in (-\infty, 1) \setminus \{0\}$. The investor seeks to maximize the expected utility of their terminal wealth, subject to constraints on the range of values of the terminal wealth.

Define the constant $A \coloneqq \theta/(\sigma(1-\gamma))$ and the process

$$Z(t) = \exp\left(\left(r + \theta\sigma A - \frac{1}{2}\sigma^2 A^2\right)t + \sigma A W(t)\right), \quad \forall t \in [0,T].$$

Problem with a Lower and an Upper Bound.

The problem with only an upper bound U was introduced and solved in Donnelly et al. (2015a). Here we extend the problem to include a lower bound $L \in (0,U)$, below which the terminal wealth must not fall. Combined with the upper bound U, this means that the investor's terminal wealth lies in the range $[L,U]$.

The addition of a lower bound has already been well studied in the literature (Basak 1995) and this is logical as pension savers are usually afraid of their wealth falling below a certain minimum level.

In order to avoid both the uninteresting case that the investor can immediately be assured of maximizing the terminal utility and the breaching of the non-arbitrage condition, we assume that $L < \left(x_0 + g(0)\right)e^{rT} < U$.

Problem 1. *Find* $\pi^\theta \in \mathcal{A}$ *such that*

$$\mathbb{E}\left(v(X^{\pi^\theta}(T))\right) = \sup_{\pi \in \mathcal{A}} \{\mathbb{E}\left(v(X^{\pi}(T))\right)\},$$

and $X^{\pi^\theta}(T) \in [L,U]$, almost surely.

The optimal terminal wealth with a lower and upper bound. The next proposition gives an expression for the optimal terminal wealth for Problem 2, when there is both a lower and upper bound constraint on the terminal wealth.

Proposition 1. A solution to the restricted problem at the terminal time T is

$$\begin{aligned} X^\theta(T) = \quad & (z_0 + g(0))Z(T) - \max\{0, (z_0 + g(0))Z(T) - U\} \\ & + \max\{0, L - (z_0 + g(0))Z(T)\}, \end{aligned}$$

with $z_0 > 0$ chosen so that the budget constraint is satisfied with equality by X^θ, given the investor's initial wealth $X^\theta(0) = x_0$, and savings plan g.

Proof. The proof is an adaption of the proof of a proposition in Donnelly et al. (2015a).

Proposition 2. An optimal investment strategy for Problem 2 is to invest the amount

$$\pi^{\theta}(t) := A\left[1-\Phi(d_{+}\left(t, P(t); U\right)-\Phi(-d_{+}\left(t, P(t); L\right))\right]P(t)$$

in the risky stock and the amount $X^{\pi^{\theta}}(t)-\pi^{\theta}(t)$ in the risk-free bond, in which $P(t)=\left(z_0+g(0)\right)Z(t)$ and the function d_{+} is defined for each K>0 by

$$d_{+}(t, y; K) := \frac{1}{\sigma A\sqrt{T-t}}\left(\ln\left(\frac{y}{K}\right)+\left(r+\frac{1}{2}\sigma^2 A^2\right)(T-t)\right), \quad \forall y > 0.$$

Proof. The proof follows trivially from the previous results. Details can be found in Donnelly et al (2015*b*).

***Lemma* 1.** (p-quantiles). Suppose an investor has initial wealth $x_0 > 0$ and follows the savings plan g. Define

$$\beta_p := \sigma A\sqrt{T}\Phi^{-1}(p)+\left(r+\theta\sigma A-\frac{1}{2}\sigma^2 A^2\right)T.$$

If the investor follows the optimal constrained strategy, that is the terminal wealth is constrained to lie in the range $[L,U]$, then the p-quantile of the investor's terminal wealth $X^{\theta}(T)$ is

$$Q_p\left(X^{\theta}(T); (L,U)\right) = max\left\{L, min\left\{U, (z_0+g(0))e^{\beta_p}\right\}\right\}.$$

Proof. The proof can be found in Donnelly et al (2015*b*).

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Endnotes

[1] Following the notation in the Appendix, we fix the parameter values: $r = 0, \quad \mu = 0.0343, \quad \sigma = 0.1544, \quad A = 1, \quad T = 30, \quad g \equiv 0, \quad x_0 = 300.$ Note that the choice of the parameters implies that the investor's risk aversion constant is $\gamma = -0.44$. We use parameters similar to those proposed in Donnelly at al. (2015a) and Guillén et al. (2014).

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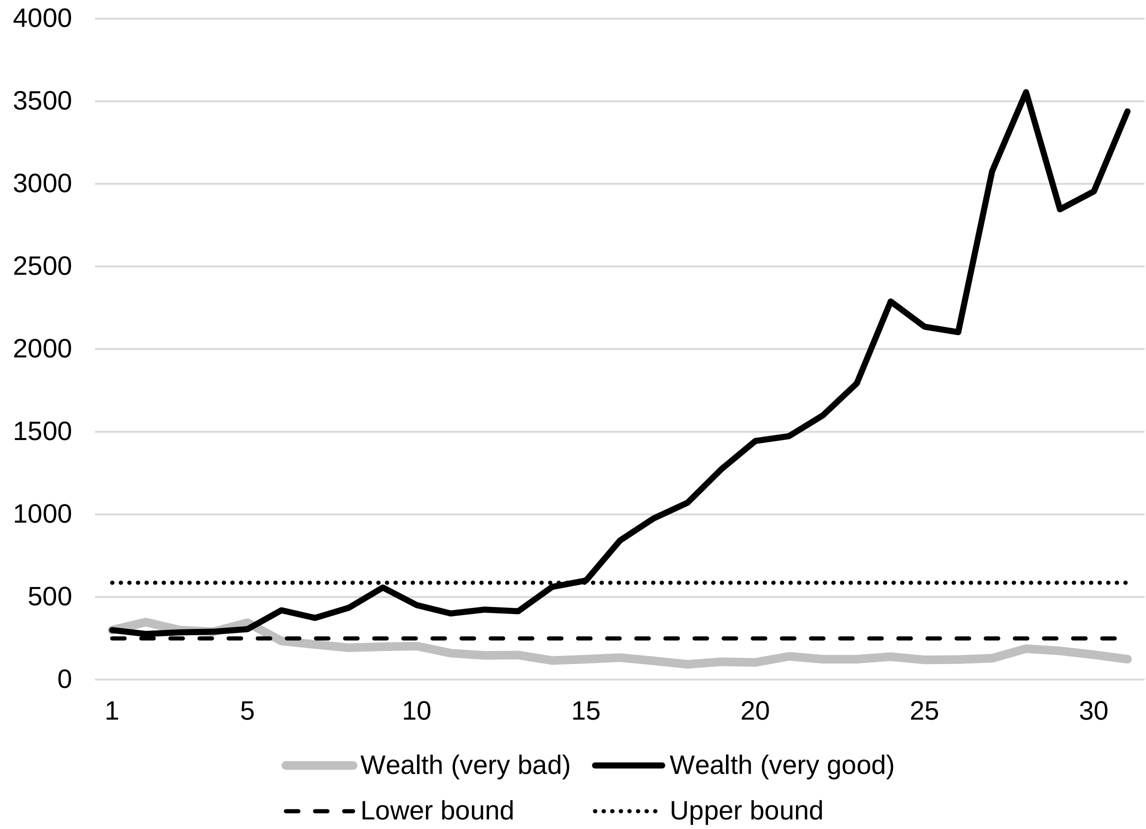

Figure 9.1. Sample paths of accumulated wealth for a one-off contribution during the 30 years for which it is invested.

Note: The strategy followed does not incorporate any bounds on the wealth at retirement. The light grey path shows a pension saver who ends with less than the initial contribution. The black path shows a pension saver who accumulates large gains. The horizontal dotted and dashed bounds are included to allow comparison with Figure 9.2.

Source: Authors' computations.

Figure 9.2. Sample paths of accumulated wealth for a one-off contribution during the 30 years for which it is invested, with bounds.

Note: Here the strategy incorporates both a lower bound (dashed horizontal line) and an upper bound (dotted horizontal line) on the wealth at retirement. The light grey path shows a pension saver who accumulates the minimum amount at retirement, which was set equal to the chosen lower bound. The dark grey path shows a pension saver who accumulates the maximum level, as fixed by the upper bound. The black path shows a pension saver who accumulates wealth in between the fixed bounds.

Source: Authors’ computations.

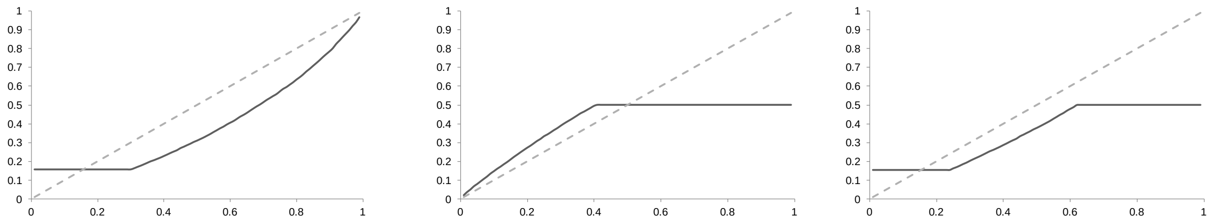

Figure 9.3. Wealth distribution quantile plots after 30 years.

Note: The dashed diagonal line corresponds to the unrestricted case. The left-most plot considers only a lower bound (solid black line), the middle plot considers only an upper bound (solid black line) and the right-most plot includes both upper and lower bounds (solid black line).

Source: Authors' computations.

Table 9.1. Distribution of wealth after 30 years for an initial investment of 300 units and various choices of the terminal lower and upper bounds.

Probability (%)	Unrestricted (no bounds)[a]	Only lower bound equal to 250[b]	Only upper bound equal to 587.10[c]	Both bounds, lower equal to 250 and upper equal to 587.10[d]
1	82.09	250.00	100.86	250.00
2.5	111.91	250.00	137.49	250.00
5	146.08	250.00	179.48	250.00
10	198.62	250.00	244.03	250.00
20	288.14	250.00	354.02	250.00
25	331.88	250.00	407.76	256.80
30	376.80	250.00	462.95	291.56
40	473.87	312.82	582.22	366.68
50	587.10	387.57	587.10	454.28
60	727.38	480.17	587.10	562.82
70	914.77	603.88	587.10	587.10
75	1038.57	685.60	587.10	587.10
80	1196.24	789.69	587.10	587.10
90	1735.38	1145.59	587.10	587.10
95	2359.53	1557.63	587.10	587.10
97.5	3080.05	2033.27	587.10	587.10
99	4198.77	2771.78	587.10	587.10

Notes:

[a] This column indicates the maximum level of terminal wealth that is obtained with the probability in the first column and no restriction on the terminal wealth.

[b] This column indicates the maximum level of terminal wealth according to the optimal strategy that restricts terminal wealth to be no smaller than 250.

[c] This column indicates the maximum level of terminal wealth according to the optimal strategy that restricts terminal wealth to be no larger than 587.10.

[d] This column indicates the maximum level of terminal wealth according to the optimal strategy that restricts terminal wealth to lie between 250 and 587.10.

Source: Authors’ computations.