Dynamic Asset Allocation with Regime Shifts and Long Horizon CVaR-Constraints

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Keywords: portfolio choice, risk management, downside risk

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JEL Classification: G11, G17

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1 Introduction

Individual investing for retirement has become increasingly important, as social security systems all over the world are forced to cut unfunded pay-as-you-go benefits due to the demographic change. Consequently, many countries have started social reforms that encourage voluntary individual investing for retirement, which on the one hand helps individuals to gain more control over their pension assets, but on the other hand exposes them to higher risk from capital markets. To mitigate risks from unsophisticated investment decisions, excessive risk taking, or extreme market movements, individual pension or retirement accounts are often augmented with a guarantee that provides individuals with a minimum level of benefits. Downside risk protection, however, is not costless, and the crucial questions are: who pays for the protection, what does the protection cost, and how to implement an effective protection? Especially in the aftermath of the 2008 financial crisis, private households with individual retirement accounts were hit hard. The sharp increase in stock volatility and the downturn of worldwide stock markets depleted the available risk budget strongly. Furthermore, interest rates went down and continue to stay at historic record lows. Such extreme movement has to be incorporated in the proper evaluation and design of individual retirement investment strategies with minimum benefit targets.

In this paper we apply a dynamic asset allocation approach to tackle these questions. We solve for both, the optimal unconstrained as well as the optimal Conditional Value at Risk (CVaR) constrained policy in a model with regime shifts in the stock volatility and the CIR-style short rate process. Thereby, the CVaR constraint is core to a risk management scheme that monitors the investor's ability to meet a money back guarantee with high confidence and restricts her asset allocation if necessary. Costs for the money back guarantee are measured as certainty equivalent welfare losses from comparing the constrained against the unconstrained policy.

Our approach is motivated by the German Retirement Savings Act from 2002 (Altersvermögensgesetz (AVmG)), though not restricted to it. Instead, we focus on its fundamental principle with respect to identifying eligible investment solutions, which can be described as an integrated investment and risk management scheme with conditional solvency requirement. The private investor chooses a certified money manager, who invest on her behalf. To qualify for a certification, the money manager has to provide a nominal money back guarantee on the invested capital. Apart from this, there are no further restrictions in place for the applied investment policy and the asset class selection. Key to this framework is a continuous risk assessment that monitors whether the current wealth level is sufficient to back the guarantee with a high degree of confidence.[1] Furthermore, no upfront guarantee costs are charged at the beginning or during the life of the contract. Instead, the investment company has to step in with its own capital, whenever the risk assessment signals an underfunding of any contract.[2] Note, in contrast to other pension schemes or related

[1]In this framework, the money back guarantee constitutes a legally binding liability. Throughout this paper, however, we use the term money back guarantee synonymously for our constrained portfolio policy, which protects the target with high confidence, but not with certainty.

[2]In this context, Gründl, Nietert, and Schmeiser (2004) coined the term contingent equity, because equity is not set aside at underwriting, but provided whenever the plan is underfunded. The authors, however, point out that the investment company has the option to default, whenever this is more profitable.

insurance contracts, individual retirement accounts cannot benefit from pooling effects, but each guarantee has to be granted separately, which increases the costs.

Previous research on minimum return guarantees are mainly carried out in an option based framework, e.g. Bodie (1995), Gründl et al. (2004), and Lachance and Mitchell (2003). In this paper, we focus on an asset allocation approach instead.[3] In the context of the German retirement savings act, Maurer and Schlag (2002) and Schnabel and Seier (2003) report that typical dynamic asset allocation schemes yield rather low costs for the investor. Maurer and Schlag (2002) use a risk-return framework to measure the costs, while Schnabel and Seier (2003) apply a utility based framework to measure the difference in certainty equivalent. Both studies benchmark several dynamic asset allocation schemes against an unconstrained base case portfolio using a standard Black and Scholes (1973) asset model.

This practical application, however, relates to more general research questions about portfolio choice under downside constraints, too. We consider a standard power utility investor who faces an exogenous downside risk constraint in a multi-period framework. Reasons for exogenous downside constraints, or risk management, are manifold. They can be found in the regulation of banks, insurance companies, pension plans, and, like in our case, investment products. The subtle differences lies in the way the downside risk is formulated in a multi-period setting.[4] For instance, Basak and Shapiro (2001) consider optimal trading between the dates of risk evaluation, while Yiu (2004) considers optimal trading with a dynamic risk constraint attached. In our framework, the downside risk constraint is defined over terminal wealth but also regularly evaluated during the course of investing.[5] The breakdown of the risk horizon for intermediate evaluation and trading is similar to the case of standard Constant Proportion Portfolio Insurance (CPPI) proposed by Black and Jones (1987), where the wealth level must always reside above a floor, typically the discounted strike, in order to meet the target at maturity. The difference is that the standard CPPI only considers two assets, a risky and a risk-free one. Optimal allocation within the risky portfolio is not covered, as changes in the risky portfolio would alter the multiplier. In our framework, we consider both, the optimal investment solution as well as the risk management scheme.[6] As a consequence, a dynamic investor anticipates the impact of risk management and adjusts her risky allocation accordingly. Furthermore, our approach also considers term structure risk, which impacts the floor quite differently depending on the remaining maturity. This is important for long-term CVaR constraints, because the CPPI generally neglects volatility in

[3] The disadvantage of the option approach is that put options with long maturities are expensive, they are thinly traded—if at all, the costs are incurred upfront, and option prices do not necessarily reflect economic costs (Lachance and Mitchell, 2003).

[4] Research on downside risk constraints in a one-period setting includes works of Roy (1952), Alexander and Baptista (2002), and Alexander, Baptista, and Yan (2007) among others.

[5] This topic also relates to the research on dynamic risk measures, because standard risk measures like VaR or CVaR are designed for static settings (Artzner, Delbaen, Eber, Heath, and Ku, 2002; Riedel, 2004; Hardy and Wirch, 2004).

[6] By contrast, related studies directly incorporate downside risk measures in the objective function (Harlow, 1991; Jarrow and Zhao, 2006; Emmer, Kluppelberg, and Korn, 2001; Alexander and Baptista, 2002) or consider loss aversion in the utility function (Barberis and Huang, 2001; Berkelaar, Kouwenberg, and Post, 2004; Fortin and Hlouskova, 2011). This alters the optimal solution, as these investors endogenously dislike downside risk. The difference to our approach is that these investor do not suffer from downside protection, while the constrained power utility investor may incur welfare losses.

the discounted floor. Hence, the downside constraint is affected by the investment decision as well as the volatility of the floor. The investor can partially hedge the latter by managing the duration of her bond investment.

We contribute to the existing literature along two directions. First, instead of relying on the common log-normal distribution, we put much effort in modelling our asset model time-varying with discrete regime switches in the stock return process and in the CIR-style short rate process. The asset model is capable of generating skewed and leptokurtic asset returns—commonly known as heavy tails.[7] Especially, time horizon effects of downside risks differ distinctively in a regime switching model when compared to a common log-normal distribution (Guidolin and Timmermann, 2006). Thus, our CVaR estimates result from a conditional distribution and can not be derived by closed form solution like under a log-normal assumption. Instead they must be evaluated path-wise, which increases the computational burden considerably. The risk constraint depends not solely on the evolution of the wealth process, but it will be affected by the short rate level as well as the stock and short rate regime. This allows us to study short-term market turbulences on interim risk evaluation dates. Second, we measure the guarantee costs against the optimal unconstrained policies for lump sump investments as well as savings plans. In contrast, previous works have reported costs against ad-hoc defined benchmarks but have not considered the optimal asset allocation decision of the unconstrained investor, which should be the natural basis for comparison. Furthermore, standard portfolio insurance strategies like the CPPI cannot deal with dynamic portfolio strategies, as they assume a constant risky portfolio mix until maturity and ignore interest rate risk.

Our results show that for lump sum investments, the costs of money back guarantees are negligible for very risk averse investors, but not for less risk averse investors who can suffer from such guarantees—especially in a low interest rate environment. In practice most plans are designed as saving plans, in which the investors contribute a specific amount each period instead of investing all at the beginning. In this setting, costs from the money back guarantee are somewhat smaller. For individual retirement accounts, these results are supportive, as they back evidence for low economic costs arising from money back guarantees and low welfare losses from applying simpler strategies, even in stressed market situations. On the other hand, the money back guarantee can resolve many incentive problems that arise in practice. In Germany, for instance, financial regulators are not required to approve complex investment strategies offered by the plan providers, because the providers have to step in with their own capital, whenever a plan is underfunded; this shuts down excessive risk taking effectively. Furthermore, in this framework, the money back guarantee requires no initial premium in contrast to an insurance-like option. This eases the transition from a pure public pay-as-you go pension system to a more privately funded one, because no massive upfront risk capital needs to be set aside. In fact, if capital markets developed favourably no protection costs would be ever incurred, because the investor could pursue the optimal unconstrained policy.

The paper is structured as follows. The second section presents our asset model with

[7] See Ang and Bekaert (2002b); Ang, Bekaert, and Wei (2008); Bansal and Zhou (2002); Dai, Singleton, and Yang (2007); Driffill, Kenc, Sola, and Spagnolo (2009) for evidence in regime switches in short rates and Chunhachinda, Dandapani, Hamid, and Prakash (1997); Peiro (1999) for skewness in financial returns.

discrete regime shifts in the stock and the short rate process. We also discuss how to calibrate and to simulate such regime switching models. The third section describes the investor's decision problem and the risk management scheme. The fourth and the fifth section present the asset allocation results and the welfare analysis for the unconstrained and the constrained lump sum investments, while the sixth section presents the results for savings plans. The last section concludes.

2 A Regime Switching Asset Model

2.1 Modelling Regime Shifts in Stock and Bond Returns

Our asset model is driven by discrete shifts in regimes that affect the stock return process as well as the short rate process. A similar model was applied by Ang and Bekaert (2002a) who study the impact of regimes shifts in an international asset allocation framework. They, however, only model the short rate, whereas we derive from the short rate process the whole term structure, which we use for mainly two reasons: first, we generate bond returns for investment purposes, and second, we discount future liabilities with respect to the remaining maturity.

Analogously to Hamilton (1989, 1990), the regime shifts of each process are driven by a discrete two-state Markov process, which prevails either in regime 0 or regime 1. The probability p_{ij} of a transition from regime i to regime j depends only on the current regime i. The transition probability matrix is

$$\mathbf{F} = \begin{bmatrix} p_{00} & p_{10} \\ p_{01} & p_{11} \end{bmatrix}. \tag{1}$$

We assume that the investor observes the current regime i, while the econometricians has to filter the unobservable regimes from the data.

To preserve flexibility and parsimony, we allow the two regimes of the term structure model and the stock return process to be governed by two independent processes. The joint regime s_t is then described by the combination of the individual regimes s_t^x for the short rate process and s_t^r for the stock return process. The resulting four-state joint regime s_t equals

$$s_t = s_t^x \otimes s_t^r = \begin{bmatrix} s_t^x = 1, & s_t^r = 1 \\ s_t^x = 1, & s_t^r = 2 \\ s_t^x = 2, & s_t^r = 1 \\ s_t^x = 2, & s_t^r = 2 \end{bmatrix}. \tag{2}$$

This specification allows both processes to be affected by different regime shifts but reduces the parameter space significantly. We also assume uncorrelated shocks between the CIR- and the stock return process.

The Markov switching CIR-style term structure model (MS-CIR) is in line with Bansal and Zhou (2002) and Driffill et al. (2009) with the discrete-time version of the state variable process given by

$$x_{t+1} - x_t = \kappa(s_{t+1}) \left[\theta(s_{t+1}) - x_t\right] + \sigma(s_{t+1})\sqrt{x_t}u_{t+1}, \tag{3}$$

where x_t denotes the state variable, θ the process's long-run mean, and κ the adjustment speed at which the process reverts to its long-run mean. The conditional variance of the process is $\sigma^2 x_t$, where σ serves as a scaling parameter. All parameters may depend on the realized regime s_{t+1}. The random shock $u_t \sim N(0,1)$ is independently and standard normally distributed, and hence the process conditionally normally distributed. Note, with a single state the standard Cox, Ingersoll, and Ross (1985) model ensues in the limit (Sun, 1992).

The market price of risk λ is also modelled regime-dependent with the corresponding pricing kernel

$$M_{t+1}(s_{t+1}) = \exp\left[-r_t^f - \left(\frac{\lambda(s_{t+1})}{\sigma(s_{t+1})}\right)^2 \frac{x_t}{2} - \frac{\lambda(s_{t+1})}{\sigma(s_{t+1})}\sqrt{x_t}u_{t+1}\right] \tag{4}$$

and the one-period risk free rate r_t^f. Bond prices with maturity τ are assumed to have the form

$$P_t^\tau(s_t) = \exp\left[-A_\tau(s_t) - B_\tau(s_t)x_t\right], \tag{5}$$

with $A(\tau)$ and $B(\tau)$ being deterministic functions of the maturity. No-arbitrage bond prices are imposed by the fundamental pricing equation

$$P_t^\tau(s_t) = E_t\left[M_{t+1}(s_{t+1})P_{t+1}^{\tau-1}(s_{t+1})\right]. \tag{6}$$

The constants A and B are derived by assuming joint log-normality in the process of the pricing kernel M_{t+1} and bond prices. Taking logs of (6) yields

$$\begin{aligned} \log P_t^\tau(s_t) = \;& E_t\left[\log M_{t+1}(s_{t+1}) + \log P_{t+1}^{\tau-1}(s_{t+1})\right] \\ & + \tfrac{1}{2} Var_t\left[\log M_{t+1}(s_{t+1}) + \log P_{t+1}^{\tau-1}(s_{t+1})\right]. \end{aligned} \tag{7}$$

The resulting expression for A and B for each regime i is then given by

$$\begin{aligned} A_\tau(i) &= p_{ii}\left(A_{\tau-1}(i) + B_{\tau-1}(i)\kappa_i\theta_i\right) \\ &\quad + p_{ij}\left(A_{\tau-1}(j) + B_{\tau-1}(j)\kappa_j\theta_j\right) \\ B_\tau(i) &= p_{ii}\left((1-\kappa_i-\theta_i-\lambda_i) - \tfrac{B_{\tau-1}^2(i)}{2}\sigma_i^2 + 1\right) \\ &\quad + p_{ij}\left((1-\kappa_j-\theta_j-\lambda_j) - \tfrac{B_{\tau-1}^2(j)}{2}\sigma_j^2 + 1\right), \end{aligned} \tag{8}$$

with boundary conditions $A(0) = 0$ and $B(0) = 0$. Finally, the bond yields y_t^τ are obtained by

$$y_t^\tau(s_t) = -\frac{\log P_t^\tau(s_t)}{\tau} = \frac{A_\tau(s_t)}{\tau} + \frac{B_\tau(s_t)}{\tau}x_t. \tag{9}$$

Similarly, the distribution of stock returns is driven by a Markov process, too. We assume the log excess returns r_t^e to be conditionally normally distributed with regime dependent volatility v but regime invariant mean μ. Hence, we filter the mean-adjusted process of stock excess returns to obtain the regime dependent volatility. For the regime invariant mean, we assume an expected excess return of 7%, which is in line with the long-run estimates reported in Mehra and Prescott (2003).[8] Applying a regime invariant mean reflects the common difficulties in deriving reliable estimates for the equity premium due to its large standard errors, especially in the case of regime switching models.[9]

The estimated process is described by:

$$r^e_{t+1} = \mu + v(s_{t+1})e_{t+1}, \tag{10}$$

where $e_t \sim N(0,1)$ is the random shock in stock returns.

2.2 The Data

The investor allocates her wealth among stocks, default-free government bonds, and a risk-free asset. Monthly excess stock returns are computed from returns including dividends on the value weighted CRSP-index of stocks traded on the NYSE, AMEX, and NASDAQ less the 1-month Treasury bill; both continuously compounded. Monthly returns from government bonds with maturity of five years are derived from a CIR-style regime switching interest rate model.

The interest rate model is calibrated with the 3-month, the 6-month, and the 5-year yield. The 3-month and the 6-month yields are taken from the Treasury bills provided by the Federal Reserve Bank whereas the 5-year bond yields are the Fama-Bliss yields provided by the CRSP database. In total, the data set comprises 457 monthly bond yields and index prices from December 1969 to December 2007.

2.3 Calibration of the Regime Switching Asset Model

We proxy the actually latent state variable x_t by an observable yield. Here we rely on the 3-month Treasury bill as proxy for the state variable x_t, because Treasury bills with shorter maturity are usually distorted by short-term liquidity needs.

To obtain monthly parameter estimates for the MS-CIR and the regime dependent market price of risk $\lambda(s_t)$, we assume that bond yields for a collection of maturities, specifically the 6 month and 5 years yield, are measured with error

$$y^\tau_t = \frac{A_\tau}{\tau} + \frac{B_\tau}{\tau} x_t + \iota_t. \tag{11}$$

This allows us to jointly estimate the MS-CIR model parameters and the market price of risk.[10]

[8] Mehra and Prescott (2003) estimate a nominal equity premium of 6.92% for the subperiod of 1889-2000 and a nominal equity premium of 8.4% for the subperiod 1926-2000. Due to the more or less flat US stock markets in the first decade of the 21th century, one might argue that the equity premium should be set lower. Nevertheless, we are analyzing long-term investments, and weak stock markets over a decade can also be observed in the last century, e.g. between 1970 and 1979. Hence, we stick to the long-run estimate of Mehra and Prescott (2003).

[9] Ang and Bekaert (2002a) estimate different specifications of regime switching model including a mean-invariant version. They conclude that the model can not be rejected with respect to various model selection criteria. See Vo and Maurer (2012) for an analysis of asset allocation under regime dependent expected returns and parameter uncertainty.

[10] Assuming measurement errors for a collection of yields is a commonly applied approach when calibrating term structure models, see e.g. Pearson and Sun (1994); Chen and Scott (1995); Duffie and Singleton (1997)

The first step is to infer the hidden Markov process that governs our regimes from the historical data. For this purpose we apply the Hamilton filter (Hamilton, 1989, 1990), which requires the conditional probability density given the prevailing regime. Following Driffill et al. (2009) and applying pseudo-maximum likelihood estimation, the density is assumed conditionally normal

$$P(z_t|s_t, \mathbf{z}_{t-1}, \Theta) = \frac{1}{2\pi|\mathrm{V}_{\mathrm{s_t}}|^{1/2}} e^{\left(-\frac{1}{2}\epsilon_t' \mathrm{V}_{\mathrm{s_t}}^{-1} \epsilon_t\right)} \times \frac{1}{2\pi|\Sigma_{s_t}|^{1/2}} e^{\left(-\frac{1}{2}\eta_t' \Sigma_{s_t}^{-1} \eta_t\right)}, \tag{12}$$

conditioned on the regime s_t, the observed data set $\mathbf{z_t} = \{z_t, z_{t-1}, \ldots, z_1\}$ with $z_t = (y_t^6, y_t^{60}, r_t)'$, and the parameter space

$$\Theta = \{\kappa_0, \kappa_1, \theta_0, \theta_1, \sigma_0, \sigma_1, \lambda_0, \lambda_1, \mu_0, \mu_1, \Sigma_0, \Sigma_1\}.$$

The first density corresponds to the stock return and the short rate process with the residual

$$\epsilon_t = \begin{bmatrix} x_t & - & x_{t-1} - \kappa(s_t)[\theta(s_t) - x_{t-1}] \\ r_t & - & \mu \end{bmatrix} \tag{13}$$

and its regime dependent covariance matrix

$$\mathrm{V}_{s_t} = \begin{bmatrix} \sigma^2(s_t) x_{t-1} & 0 \\ 0 & \upsilon^2(s_t) \end{bmatrix}. \tag{14}$$

The second corresponds to the measurement errors on the yields with 6 month and 5 year maturity

$$\eta_t = \begin{bmatrix} y_t^6 & - & \frac{A_6(s_t)}{6} & - & \frac{B_6(s_t)}{6} x_t \\ y_t^{60} & - & \frac{A_{60}(s_t)}{60} & - & \frac{B_{60}(s_t)}{60} x_t \end{bmatrix} \tag{15}$$

with regime dependent covariance matrix

$$\Sigma_{s_t} = \begin{bmatrix} \sigma_{6m}^2(s_t) & \sigma_{6m,5y}(s_t) \\ \sigma_{6m,5y}(s_t) & \sigma_{5y}^2(s_t) \end{bmatrix}. \tag{16}$$

We obtain parameter estimates by maximizing the likelihood in equation (12) numerically from multiplee different starting values. Using different sets of starting values reduces the risk of being trapped in a locally optimal solution. The parameter estimates with the highest likelihood among the trials are reported in Table 1 with numerically derived standard errors. The asset model is used to simulate sample paths of monthly stock returns, bond returns, and interest rates. Based on these simulations, we evaluate the CVaR of future wealth and solve for the optimal portfolio policies using Monte Carlo integration.

[Table 1 about here.]

We label the regimes in the stock return (S) and CIR process (C) as high volatility (HV) and low volatility (LV) regime, respectively.[11] This yields in total four possible regimes:

[11] Actually, the volatility of the short rate process is $\sqrt{x_t}\sigma(s_t)$, but for convenience we treat the regime dependent scaling parameter $\sigma(s_t)$ and the process's regime dependent volatility synonymous. Whenever necessary to avoid ambiguity, we will clearly state in the text, whether the scaling parameter or the process's standard deviation is meant.

SLV, CLV, SHV, and CLV. In the case of the stock return process, it is by construction only the volatility that differentiates the regimes. In short rate process, however, regime shifts affect all parameters. The short rate's low volatility regime exhibits a slightly lower market price of risk (-0.0122) and a lower long-run mean (0.0023) compared to the high volatility regime (-0.0089 and 0.0071). The lower market price of risk indicates a slightly steeper yield curve in the low volatility regime. Furthermore, the MS-CIR's low volatility regime is more persistent than the high volatility regime, which can be seen by the transition probabilities. Conditioned on being in the low (high) volatility regime it is relatively more (less) likely to stay in this regime (0.9787 vs. 0.9080). The long-run probabilities show that the system prevails in the low volatility regime by about 81%. Hence, the short rate process exhibits rather short periods of higher volatility. Similarly, the stock return process prevails about 75% of the time in the low volatility regime.

A long-term investor may be less affected by these short periods of higher volatility in both regimes, since the system quickly moves to its steady state. In light of short-term risk constraints, however, the impact of those periods of excess volatility, though potentially short-lived, can be of great importance. For instance, an investor with a downside risk constraint on terminal wealth might be tempted to consider the long-run distribution only. However, evaluation of the risk constraint from period to period during the investment horizon will exhibit potentially large swings in her risk measurement.

2.4 Simulation of the Regime Switching Asset Model

We use Monte Carlo simulation to generate the asset return paths, which will be used to evaluate the CVaR of future wealth and the expected utility in the portfolio optimisation problem later on. Stock returns are generated from a standard Euler-scheme

$$r^e_{t+1} = \mu + v(s_{t+1})Z_v \tag{17}$$

where r^e_{t+1} is the continuously compounded stock excess return, μ its unconditional mean, $v(s_{t+1})$ its regime dependent volatility, s_{t+1} the prevailing regime, and Z_v a standard normal random variable. Hence, the stock return distribution is a mixture of normals, whereby the current regime is first determined by a binomial random variable with probability $P(s_{t+1}|s_t)$.

The short rate's square-root process, however, requires a different sampling scheme, because its transition probability is not normal but non-central chi-squared. Unfortunately, sampling from a non-central chi-square process is computationally demanding. Therefore, Andersen (2007) proposed a quadratic exponential (QE) simulation algorithm for approximating square-root processes, which we adopt here.[12] The main problem arises from the non-negativity condition of the square-root process. Hence, simply sampling from the Euler-scheme in (17) is likely to result in negative short rates, whenever the short rate is very small. Therefore, the rationale of the QE scheme is to match moments by using a quadratic scheme, whenever the short rate x_t is sufficiently large and an exponential scheme otherwise.

[12]Originally, Andersen (2007) developed the sampling scheme for conditional volatility models (Heston, 1993), but the same rationale for square root processes applies here.

The conditional mean m and volatility σ of equation (3) are given by[13]

$$m = \theta + (x_t - \theta)e^{-\kappa\Delta} \tag{18}$$

and

$$s^2 = \frac{x_t \sigma^2 e^{-\kappa\Delta}}{\kappa}\left(1 - e^{-\kappa\Delta}\right) + \frac{\theta\sigma^2}{2\kappa}\left(1 - e^{-\kappa\Delta}\right)^2. \tag{19}$$

For sufficiently large values of the short rate x_t, the quadratic scheme draws from

$$x_{t+1} - x_t = a + (b + Z_v)^2 \tag{20}$$

where Z_v is a standard normal random variable. a and b are parameters that are determined by moment matching. They depend on the time step Δ, on x_t, and on all other parameters of the short rate process.

To define a and b^2, set $\psi = s^2/m^2$, then for $\psi \leq 2$

$$b^2 = 2\psi^{-1} - 1 + \sqrt{2\psi^{-1}}\sqrt{2\psi^{-1} - 1} \geq 0 \tag{21}$$

and

$$a = \frac{m}{1 + b^2}. \tag{22}$$

Andersen (2007) shows that if $\psi \leq 2$, which is the case for sufficiently large values of x_t, the moments of the quadratic scheme equates to the true moments m and s^2.

For low values of x_t, or more precisely if $\psi > 2$, the quadratic scheme performs poorly and the moment matching fails.[14] Hence, an alternative exponential sampling scheme is applied for low values of x_t. Andersen (2007) proposes an approximative probability density for low values of x_t, where the probability mass in the origin can be specified explicitly. The resulting cumulative distribution function is

$$\Psi(z) = Pr(\Delta x_{t+1} \leq z) = p + (1 - p)\left(1 - e^{-\beta z}\right), \qquad z \geq 0, \tag{23}$$

where $\beta \geq 0$ and $p \in [0, 1)$ are both parameters to be determined. The inverse of equation (23) is

$$\Psi^{-1}(u; p, \beta) = \begin{cases} 0, & 0 \leq u \leq p, \\ \beta^{-1}\frac{1-p}{1-u} & p < u \leq 1 \end{cases}. \tag{24}$$

Thus, variates for low x_t values can be sampled by the inverse distribution function method

$$\Delta x_{t+1} = \Psi^{-1}(U_v; p, \beta), \tag{25}$$

where U_v is a uniform random variable. Analogous to the quadratic scheme, the moments of the exponential scheme will match the true moments m and s^2 only if $\psi \geq 1$.[15] p and β are defined as

$$p = \frac{\psi - 1}{\psi + 1}, \in [0; 1) \tag{26}$$

[13]For better readability, we omit the regime dependent notation. Note, that in the short rate process, all parameters are regime dependent, and the formulas apply accordingly for each regime.

[14]Recall that negative nominal interest rates are in general not meaningful and also strictly prohibited in the square-root process.

[15]More precisely, the system of equations has a single solution if $p < 1$, but the stated density function additionally requires $p > 0$. Both is the case for $\psi \geq 1$ (Andersen, 2007).

and

$$\beta = \frac{1-p}{m} = \frac{2}{m(\psi+1)} > 0. \tag{27}$$

Finally, we need a switching rule for both sampling schemes. The quadratic scheme is applicable if $\psi \leq 2$, while the exponential scheme requires $\psi \geq 1$. Fortunately, both intervals overlap such that always at least one of both sampling schemes is valid. According to Andersen (2007), we find that the sampling results are quite robust to the exact choice of the threshold for ψ so that we set it to 1.5.

3 The Investor's Problem

3.1 The Optimal Unconstrained Policy

Consider first the optimal policy for an unconstrained investor with constant relative risk aversion (CRRA) who faces regime shifts in the stock and short rate process. The investor optimises her utility over terminal wealth W_T and trades among a stock investment, a bond investment with fixed maturity of five years, and the one-period risk-free asset. We consider a simple constant maturity bond portfolio, because in practical applications complex liability management can be prohibitively costly on an individual retirement account basis. Five years of maturity are chosen to match the duration of standard medium-term bond portfolios. We assume the investor observes the current state of the economy, which is described by the current short rate level, the current regime of the stock return process, and the current regime of the short rate process.[16] We ignore typical frictions like taxes on capital gains or transaction costs, and also rule out short selling.

The investor solves for the optimal policy $\boldsymbol{\omega} = \{\omega_s\}_{s=t}^{T-1}$ that maximizes her terminal wealth

$$V(W_t, Z_t) = \max_{\{\omega_s\}_{s=t}^{T-1}} \mathrm{E}_t \left[u\left(W_T\right)\right] \tag{28}$$

subject to the intertemporal budget constraints

$$W_{s+1} = W_s \left(\omega_s' R_{s+1} + R^f\right), \ \forall s \geq t, \tag{29}$$

where Z_t denotes the state variables, W_t the current wealth, and R_{s+1} the one-period excess return vector of the stock and the bond investment. Note, $\boldsymbol{\omega} = \{\omega_s\}_{s=t}^{T-1}$ denotes not only a single decision, but the sequence of all optimal decisions from now until $T-1$. Fortunately this complex optimisation problem can be restated as a sequence of simpler optimisation problems via dynamic programming.

With standard power utility and risk aversion parameter γ we obtain[17]

[16] Of course, all these factors are latent and unobservable by nature. Endowing the investor with these information is likely to exaggerate the impact of regime shifts. Nevertheless, we argue that this is a first valuable step to assess the importance of regime shifts and the results serve as an upper bound estimate for utility costs.

[17] The notation is mainly in line with Brandt, Goyal, Santa-Clara, and Stroud (2005).

$$V_t\left(W_t, Z_t\right) = \max_{\omega_t} \mathrm{E}_t\left[\max_{\{\omega_s\}_{s=t+1}^{T-1}} \mathrm{E}_{t+1}\left(\frac{W_T^{1-\gamma}}{1-\gamma}\right)\right], \tag{30}$$

$$= \max_{\omega_t} \mathrm{E}_t\left[u\left(W_{t+1}\right) \underbrace{\max_{\{\omega_s\}_{s=t+1}^{T-1}} \mathrm{E}_{t+1}\left[\left(\prod_{s=t+1}^{T-1}\left(\omega_s' R_{s+1}^e + R^f\right)\right)^{1-\gamma}\right]}_{\psi_{t+1}(Z_{t+1})}\right]. \tag{31}$$

Since the CRRA utility is homothetic in wealth, the stated problem only depends on the state variables in Z. With normalized wealth $W_t = 1$ it can be reduced to

$$\frac{1}{1-\gamma}\psi_t\left(Z_t\right) = \max_{\omega_t} \mathrm{E}_t\left[u\left(\omega_t' R_{t+1} + R^f\right)\psi_{t+1}\left(Z_{t+1}\right)\right]. \tag{32}$$

The solution to this problem can be obtained by backward induction with terminal condition $\psi_T(Z_T) = 1$. Thus, in $T-1$, the investor simply solves a single period problem.

3.2 CVaR Risk Management

The most widely applied risk measure in financial risk management is certainly still the Value at Risk (VaR), a quantile risk measure defined as the loss that won't be exceeded with a certain probability. It is used by insurance companies, banks, investment companies as well as financial regulators.

The VaR_α is defined as the level of loss L that won't be exceeded with probability $1-\alpha$

$$P(L > VaR_\alpha) = (1-\alpha). \tag{33}$$

The VaR is quasi-standard and widely spread due to its intuitive and simple concept, as it reflects the perception of risk associated to regular market conditions. Nevertheless, the VaR is not a coherent risk measure in the sense of the axioms of Artzner, Delbaen, Eber, and Heath (1999)[18], it does not account for the magnitude of extreme losses exceeding itself, and it can lead to adverse portfolio policies that increase risk in the worst market conditions.[19]

By contrast, the Conditional Value at Risk (CVaR) is a coherent risk measure. It is defined for continuous distributions as the expected loss L that exceeds the VaR_α

$$CVaR_\alpha = E(L|L > VaR_\alpha). \tag{34}$$

[18]Most importantly, coherent risk measures imply subadditivity, which essentially states that they allow for diversification effects. By contrast, the portfolio VaR can actually be higher than the VaR of each individual asset. A special case is the normal distribution, where the VaR is coherent and proportional to the standard deviation.

[19]Basak and Shapiro (2001) show that VaR-style risk management schemes result in adverse portfolio policies, because the VaR accounts only for the frequency and not the magnitude of loss exceeding itself. Thus, a VaR constrained investor would only insure in states where insurance is cheap and expected losses were small, but would take the full loss whenever insurance was expensive and expected losses were large.

Furthermore, Rockafellar and Uryasev (2000) provide a representation for CVaR optimisation problems that is convex and differentiable for continuous distributions.[20] An approximation for discrete distributions is given by

$$F_\alpha(\omega, \zeta) = \zeta + \frac{1}{N(1-\alpha)} \sum_{k=1}^{N} [f(\omega, r_k) - \zeta]^+ , \tag{35}$$

where α denotes the confidence level, ω the portfolio weights, ζ the VaR_α, N the number of samples, and $f(\omega, r_k)$ the portfolio loss, which is a function of the portfolio weights ω and the k-th return vector sample. The virtue of this representation is that it is no longer necessary to determine the VaR_α first. Minimising $\text{CVaR}_\alpha(\omega)$ over ω is equivalent to minimizing $F(\omega, \zeta)$ over ω and ζ. Besides being convex and piecewise linear in ζ, this CVaR representation for discrete distributions is not differentiable with respect to ζ. Nevertheless, this problem can still be minimised by transforming it into a linear programming problem (Rockafellar and Uryasev, 2000).

Then minimising (35) is equivalent to minimising the linear expression

$$\zeta + \frac{1}{N(1-\alpha)} \sum_{k=1}^{N} u_k \tag{36}$$

subject to

$$u_k \geq 0 \qquad \text{and} \qquad f(\omega, r_k) + \zeta + u_k \geq 0 \qquad \text{for } k = 1, \ldots, N. \tag{37}$$

Note, the portfolio loss $f(\omega, r_k)$ is also linear in the portfolio weights ω. Furthermore, this CVaR representation does not imply any specific assumptions about the loss distribution but holds for general loss distributions. It can be readily applied to historical samples or to simulated samples without the need to assume a specific distribution. Hence, we can easily adopt it to our simulated asset model paths. Throughout the paper we will consider only CVaR as downside risk measure due to its favourable properties.

In our framework, the CVaR constraint is conditioned on a long-term and maturing investment horizon. At the terminal date, the investor requires a pre-defined minimum rate of return at a given confidence level. The underlying risk constraint is conditioned on the terminal wealth W_T such that

$$CVaR_\alpha(W_T) \geq \underline{W}, \tag{38}$$

where $\underline{W}$ denotes the minimum required wealth level at the end of the investment horizon.[21] Actually, the investor seeks to manage the risk of her terminal wealth only, but at each rebalancing step t, she has to monitor her current risk of missing the required minimum rate of return at the final date T. Hence, at each rebalancing step, the future liabilities are discounted by the corresponding risk-free rate to evaluate the current funding level. Let $\tau = T - t$ be the remaining investment horizon, then the current funding level is given by

$$C_t = \frac{W_t}{\underline{W} e^{-y_t^\tau \tau}}. \tag{39}$$

[20] This is not the case for the VaR, which is not convex in the portfolio weights.

[21] Note that we have defined VaR and CVaR over losses, which is common practice in risk management. Here, CVaR is applied on wealth levels, but the definitions for this case hold analogously. For instance, the VaR_α for loss distributions is its α-quantile, whereas the VaR_α for wealth distributions is its $(1-\alpha)$-quantile.

At each date t, we restrict the next period's CVaR_α to

$$CVaR_\alpha\left(\frac{W_{t+1}}{\underline{W}e^{-y_{t+1}^{\tau-1}(\tau-1)}}\right) = CVaR_\alpha(C_{t+1}) \geq 1. \tag{40}$$

Note, that this CVaR is affected by the uncertain investment outcome given by W_{t+1} but also by the interest rate risk given in the denominator. The next periods term structure, and hence the yield y_{t+1} of the bond with maturity $\tau - 1$ is stochastic as well.

The advantage of using the risk-free yield as discount rate is, that it is objectively observable, traded at the market, and hence not prone to estimation risk. Furthermore, as long as the funding level is greater than unity, the remaining liability can be fully hedged by buying the zero coupon bond that matches the liability's maturity.[22]

In some states, however, our wealth level will fall below the CVaR-constraint, and we need a pre-determined rule for such states instead.[23] For these cases, we decide to lock-in the current funding level, and thus the current shortfall, by enforcing the investor to switch into the bond with maturity that matches the remaining investment horizon. This is equivalent to realizing the shortfall immediately and consequently the money manager loses discretion about the allocation until maturity; in other words she is stopped out. Of course, this is costly in terms of utility and we expect the money manager to act strategically in order to avoid such events, e.g. by willingly reducing the allocation to stocks, even if the next period's CVaR constraint is not binding. Hence, we analyse if CVaR-style risk management schemes induce intertemporal hedging demands. For all CVaR-constrained policies, we choose a confidence level of $\alpha = 0.95$.

3.3 Numerical Solution Methods

Numerically, we solve this problem by discretizing the state space with the current regime, the current short rate, and in case of any CVaR constraint, the current funding level as our state variables. Using the terminal condition $\psi_T = 1$, we optimise the one-period problem in $T - 1$ and move backwards in time through the grid. The expected utility is solved by Monte-Carlo integration. E.g., for a one-period investment it is approximated as follows:[24]

$$\max_{\omega_t} \frac{1}{I}\sum_{j=1}^{I}\left\{\frac{\left[(1-\omega_t'\mathbf{1}_K)\exp(r_{j,t}^f) + \omega_t'\exp(r_{j,t}^f\mathbf{1}_K + r_{j,t+1}^e)\right]^{1-\gamma}}{1-\gamma}\right\}, \tag{41}$$

where I denotes the number of simulation paths, K the number of assets, and $\mathbf{1}_K$ a $K \times 1$ vector of ones. In the dynamic case, we evaluate a grid with dimension $S \times X \times C$, where

[22] Alternatively, the risk manager could discount the liability with the optimal constrained policy return. In practice, however, this requires knowledge about the conditional distribution of terminal wealth, which depends on the current state variables, but also on any future action of the investor, say her portfolio policy. This is an extremely complex as well as problematic task, because of estimation risk and potential moral hazard issues.

[23] Note, in contrast to a perfectly secured portfolio insurance, the CVaR risk management scheme allows for $1 - \alpha$ shortfalls.

[24] Other authors who also use Monte Carlo methods to approximate expected utility are Barberis (2000); Honda (2003); Guidolin and Timmermann (2007).

$S = 4$ is the number of regimes, $X = 30$ the number of short rate levels ranging from 0.5% to 15%, and $C = 40$ the number of wealth to liability ratios ranging from 1 to 5, except for risk aversion $\gamma = 2$, for which we choose a higher grid limit of 7. For funding levels less than unity, we apply a pre-defined rule. In any of those states, the investor will switch to a bond with maturity $T - t$. The value function in each state is stored and values lying in between the knots are interpolated by cubic splines; values beyond the grid limits are extrapolated.

For the optimal policies, we consider an investment period of 10 years with yearly rebalancing. The asset model, however, is simulated on a monthly base. For each of these 120 months, we draw 20,000 Sobol quasi-random numbers for each innovation process; for the regime, the stock return, and the short rate innovation. We take these simulated paths to integrate the utility expectations analogously to equation (41). Due to the high dimensionality of our problem, we use a scheme to additionally randomise the resulting sequence of Sobol numbers. Quasi-random number sequences tend to produce correlation patterns in higher dimension among the actually independent random numbers. Randomisation schemes as proposed by Matousek (1998) can reduce those correlation patterns to an acceptable level. Despite the problems in high dimensions, we still decide to use Sobol numbers to keep the optimisation tractable in a dynamic setting. For instance, we transform the CVaR-constraint into a linear constraint according to equation (36) and an excessive amount of samples would be detrimental for the optimisation routine.[25]

3.4 Measuring Welfare Costs

In the welfare analysis, we focus on two important aspects. First, we investigate the benefits of acting dynamically instead of myopically. Recent works have reported that financial literacy is an important aspect in individual retirement investment, because many individuals lack the ability to adopt most complex strategies like optimal dynamic policies.[26] Similarly, money managers may have the financial literacy to understand and implement dynamic strategies, but the infrastructure to perform such calculation for each individual account is certainly a bottleneck in practical applications. Therefore, it is important to decide if the costs of applying simpler myopic strategies truly outweighs the reduction in complexity.

Second, we investigate how costly the money back guarantee is to the investor. Recall that our framework implies a CRRA utility investor who has no intrinsic need for a risk management scheme. In fact, the constraints are set externally by the financial regulator who does not want to bear the costs of a failed private retirement scheme. In Germany, for instance, the risk management is part of the conditional solvency scheme, which prevents excessive risk taking caused by either moral hazard or bad investment decisions. In either case, the individual investor has an implicit put option on receiving social welfare if her retirement investment fails. Hence, a money back guarantee shuts down intended or unintended increases in risk taking.

[25] Nevertheless, our optimisation is not a pure linear problem, because our objective function, the CRRA utility function, is non-linear. To solve this problem, we apply a large-scale optimiser implemented by the Numerical Algorithm Group (NAG), which belongs to the class of sequential quadratic programming (SQP) optimisation algorithms.

[26] For instance, a survey by Benartzi and Thaler (1999) shows that subjects chose inconsistent asset allocations when faced with either the short-term or the long-term distribution of the same return process.

As in Ang and Bekaert (2002a), welfare costs are measured as utility loss from pursuing the suboptimal policy α^{+} instead of the optimal policy α^{*}. We can derive the wealth $\bar{w}$ that compensates an investor for using the constrained solution by solving

$$E_0\left[U(W_T^{*}|W_0=1)\right] = E_0\left[U(W_T^{+}|W_0=\bar{w})\right]. \tag{42}$$

For a CRRA investor the required wealth $\bar{w}$ can then be calculated by[27]

$$\bar{w} = \left(\frac{\psi_{0,T}^{*}}{\psi_{0,T}^{+}}\right)^{\frac{1}{1-\gamma}}. \tag{43}$$

The costs are expressed as dollar per wealth unit $w = 100 \times (\bar{w}-1)$. Hence, w is the costs per \$1 investment that is required to compensate a CRRA investor for any utility losses from the suboptimal investment policy.[28]

4 The Effect of Regime Shifts in Stock Returns and Interest Rates

4.1 Asset Allocation Results

[Figure 1 about here.]

First, we solve for the unconstrained policy, which serves as our benchmark solution against which we measure the costs of risk management. Costs are calculated as utility loss from the constrained versus the unconstrained solution. In addition to the optimal dynamic strategy, we consider two simplified strategies. The myopic strategy ignores the future impact of today's decision and instead allocates at each rebalancing date according to the optimal one-period rule. This reduces the computational burden extremely but still requires regular actions and monitoring. A further simplification is the buy-and-hold strategy, which allocates only once at the beginning of the investment period, and hence requires no future actions.[29]

The impact of regime switches and the short rate level is best illustrated by the buy-and-hold strategy. Figure 1 plots the optimal allocation to stocks for a buy-and-hold investors with different risk aversion (top to bottom panels) against the investment horizon. The left panels vary the initial short rate level and fix the initial regime at the low volatility regimes, while the right panels vary the initial regimes and fix the short rate level at its historic mean. The plotted stock fractions display the conditional optimal stock allocation for a buy-and-hold investor with remaining investment horizon given on the horizontal axis and an initial short rate level or regime represented by the different lines. For instance in the left upper graph, a $\gamma = 2$ investor allocates about 80 percent to stocks if the initial short rate level

[27] CRRA utility is homothetic in initial wealth and thus $E_0\left[U(W_T^{\diamond}|W_0=1)\right] = V^{\diamond}/(1-\gamma)$ for $\diamond = *, +$.

[28] Costs are not annualized but always given for a 10 year investment plan.

[29] Here, we assume that the buy-and-hold investor is able to consider the time horizon effects of the asset model and optimises her policy according to the $T-$th period distribution. Recall that our asset model is time-varying such that the terminal distribution of the asset returns depends on the current state and the investment period.

in $t = 0$ is high and the remaining investment period is one year. Whenever her remaining investment period is ten years, the allocation will be 100 percent. While the short rate level can and will vary over the course of the investment period, only the short rate level at the beginning in $t = 0$ is relevant. The same holds for the initial regimes in the right panels.

Both panels show that due to varying initial regimes and short rate levels the allocations move monotonically to an optimal long-term equilibrium allocation. Regime switches lead to more pronounced allocation shifts at the beginning, while changes in the short rate level result in more persistent shifts. For instance, regime switch effects are negligible for investment horizons longer than five years, irrespective of the risk aversion, while shocks in the short rate level are still in effect at these horizons.

In contrast to the buy-and-hold policy, the myopic and the optimal dynamic policies require regular rebalancing according to the prevailing states. The allocations for these strategies are displayed in Table 2 with varying risk aversions from left to right and the prevailing regimes from top to bottom. The subpanels show the optimal stock fraction for different investment periods in years (top to bottom) and for different short rate levels (left to right). The first row of each subpanel displays the 1-year allocation, which is equivalent to the myopic allocation, and the remaining rows show the intertemporal hedging demands for the longer investment periods. For instance, in the first row of the last column, we find that the optimal 1-year allocation to stocks is 21.7 percent for an investor with risk aversion $\gamma = 10$, whenever the short rate is one standard deviation above its mean ($+1\sigma_x$) and the stock as well as the CIR process prevail in the low volatility regime (SLV-CLV). The second row of the last column shows that a dynamic investor will reduce this allocation by 1.9 percent to 19.8 percent if her investment horizon is two instead of one year. By contrast, a myopic investor sticks to her 1-year allocation irrespective of the number of remaining investment periods.

[Table 2 about here.]

We report only the stock fraction, because the optimal policy is always fully invested in either stocks or bonds with no allocation to the risk-free asset; hence, the bond allocation is always 100 percent less the stock allocation. That is true for the given risk aversion parameters ($\gamma = 2, 5, 10$), the time horizons ($t = 1, \ldots, T$), the current short rate level, and the current regime.[30]

Looking at the myopic allocation in the first row of each subpanel, we find the stock fraction decreasing with the risk aversion as well as with the short rate level (left to right). Of course, more risk averse investors prefer bonds over stocks, while higher short rate level imply higher carry returns from bonds and increase their attractiveness compared to equities. The impact of the regimes (top to bottom) is more dominant for stock regimes than for short rate regimes. The allocation for all risk aversion is qualitatively similar, whenever the stock regime is in the low volatility regime (SLV-CLV and SLV-CHV) or in the high regime (SHV-CLV and SHV-CHV) and mainly invariant to the short rate regime. The short rate regime is only relevant if the short rate prevails above its mean. Then, bonds become less attractive for all risk aversions ceteribus paribus (SLV-CLV vs. SLV-CHV and SHV-CLV vs. SHV-CHV).

[30] The only exception is the myopic allocation for risk aversion $\gamma = 10$ and a short rate level one standard deviation below its mean. In this case, the investor holds about 10 percent in the risk-free asset.

The intertemporal hedging demands, which are displayed in the remaining rows of each subpanel (top to bottom), are negative for stocks but positive for bonds. This is in line with the fact that our asset model mainly induces momentum in the stock returns and mean-reversion in bond returns.[31] The magnitude for a $\gamma = 2$ investor, however, is negligible with a maximum reduction in stocks of less than 5.5 percent in the low stock/high short rate volatility state (SLV-CHV) and only at elevated short rate levels. For a high risk averse investor ($\gamma = 10$) who already holds more bonds, however, the stock allocation may drop by up to 50 percent of the myopic allocation.

4.2 Welfare Analysis

As shown in the previous section, regime shifts and interest rate risk do induce intertemporal hedging demands. However, the costs of neglecting those intertemporal hedging demands are small. Table 3 reports the cost of myopia for an unconstrained policy. For $\gamma = 2$, welfare cost are virtually zero for all states. An investor with $\gamma = 5$ loses about 12 to 24 basis points and for very risk averse investors with $\gamma = 10$, the costs are somewhat higher ranging from about 40 up to 92 basis points. This is consistent with the findings about intertemporal hedging demands in the previous section, where hedging demands increase with the risk aversion. Considering the complexity of the dynamic compared to the myopic strategies, we argue that these costs are small. Furthermore, the costs reported here are an upper bound estimate as we neglect estimation risk and allow the investor to perfectly observe the latent states of regimes and short rate levels.

Analogous to the existing literature, we find strong evidence for the importance of regime switches in asset allocation but few arguments for incorporating intertemporal hedging demands.[32]

[Table 3 about here.]

5 The Effect of Money Back Guarantees

5.1 Asset Allocation Results

So far, we have considered the unconstrained investment case under a regime switching asset model. Our objective, however, is the evaluation of a constrained strategy that ensures a nominal money back guarantee with high confidence for long-term investors. Thereby, the risk management rule is straightforward. We discount the future liabilities with the current term structure to determine the current funding level and require it to be also higher than unity the next period under any eligible allocation. Hence, the investor is required to consider two competing time horizons; on the one hand she has to meet a short-term restriction for

[31] Ang and Bekaert (2002a) highlight the momentum generating properties of regime switching stock returns and the impact of international asset allocation as well.

[32] Ang and Bekaert (2002a) report small utility losses from myopic investment in a regime switching international asset allocation framework. However, they also find that ignoring regime shifts is extremely costly.

the risk management scheme and on the other hand she has to optimize utility of her long-term terminal wealth. Similar to regime switches, CVaR-constraints are also capable of introducing intertemporal hedging demands as the investor weighs the benefit of a more aggressive allocation today versus the risk of being stopped out in later periods.

[Figure 2 about here.]

Figure 2 displays the optimal stock allocation for a myopic (left) and a dynamic (right) investor at different funding levels C_t as defined in equation (39). The risk aversion is varied from top to bottom, while the short rate level is fixed at its mean. The figure represents a sectional view of the three dimensional optimal policy grid for a selection of relevant states. It describes the policy that the investor should apply given her current state.

Recall that the myopic allocation does not account for the remaining investment periods, and hence for sufficiently large funding levels, the left panels show a constant stock allocation for all investment horizon. We found the constrained allocation to be virtually identical to the unconstrained solution, whenever the funding level is greater than 1.5, irrespective of the states or risk aversion. Therefore, it illustrates the allocation for a non-binding CVaR. For instance, at risk aversion $\gamma = 2$ the allocation is constant at funding levels greater or equal to 1.5; at risk aversion $\gamma = 5$ and $\gamma = 10$ it is already constant for funding levels greater than or equal to 1.2. The factor that drives the myopic allocation with respect to the time horizon is only the risk management rule, which becomes binding for lower funding levels. Of course, the constraint is also more binding for more aggressive investors, as they prefer riskier allocations compared to defensive investors. Nevertheless, more risk averse myopic investors with $\gamma = 5$ or $\gamma = 10$ will only adjust their allocations for horizons of at least four years and at funding levels close to unity.

The impact of the CVaR-constraint seems unintuitive at first sight, because the investor tends to *decrease* her stock allocation with increasing investment horizon. The considered CVaR estimate, however, is always a one-year estimate and is mainly determined by two factors: the investor's allocation and the duration of the liabilities. Hence, an increasing CVaR with respect to the remaining investment horizon reflects the higher interest rate risk in the liability.

The right panels display the dynamic stock allocation, which is always less than or equal to the myopic allocation. We know this result already from the previous section from our analysis of the unconstrained policy and conclude that the time-varying asset model induces decreasing stock allocation with increasing investment horizon. Similarly, the investor acts as if she were unconstrained from the risk management rule at a funding level of 1.5, and reduces her stock allocation for increasing investment periods accordingly. As the CVaR constraint becomes binding, both the myopic as well as the dynamic investor reduce their stock allocation in order to meet the restriction. Nevertheless, the dynamic investor reduces her stock allocation much stronger than the myopic investor. If unconstrained, the myopic investor always sticks to the one-period allocation irrespective of the investment horizon. Thus, any decrease in the myopic stock holdings stems only from the one-period CVaR restriction. The additional reduction in the dynamic investor's allocation compared to the myopic investor is due to the time-varying liability process and the risk of of being stopped out in the course of investing.

At low current funding levels C_t, a very risk averse myopic investor with $\gamma = 10$ and ten remaining investment years reduces her stock allocation slightly from 28 percent to 21 percent, while a dynamic investor reduces her allocation to about 8 percent. Furthermore, when comparing the dynamic solution at $C_t = 1.5$ to the myopic solution at $C_t = 1.1$, one can see that even though the virtually unconstrained policy was eligible at $C_t = 1.1$, the investor still chooses a far less risky allocation than required from the next-periods risk constraint. This highlights her additional intertemporal hedging demands arising from the risk management rule.

In summary, risk management via downside risk constraints can be considered as ex ante preventive action to control excessive risk taking as in the myopic case. Those preventive actions are one aspect of the overall hedging costs. Regularly monitoring may also detect breaches of risk constraints during the investment period, because CVaR-based risk management insures only partly. In that case, the investor shall be forced to pursue a more conservative investment scheme to reduce further downside risk and to prevent any gambling. Those rules must be formulated sufficiently restrictive to provide an incentive for the investor to avoid being trapped in those situations and losing discretion about her portfolio as seen in the dynamic case. The later aspect can be considered as punitive actions. Hence, both preventive as well as punitive aspects of a CVaR-based risk management scheme makes up for the costs of risk management.[33]

5.2 Welfare Analysis

In this section, we return to our main objective of measuring the costs of money back guarantees. Therefore, we have derived the optimal strategies with risk management in place and without. In contrast to previous findings, we measure the costs against the optimal solution and not only against some ad hoc benchmark strategies and account for different states of the economy. For instance, in the aftermath of the financial crisis in 2008, many individual retirement plans that approached maturity as well as new plans faced an extremely unfavourable situation. The sharp decline in equity markets that erased potential previous gains of existing plans was followed by a low interest rate environment with high stock market volatility.

Table 4 displays the welfare costs measured in certainty equivalents. The panels vary the risk aversion and alternative investment policies. Each subpanel presents the results for alternative initial states described by varying short rate levels and regimes. Note that the funding level is not explicitly stated in the Table; it is rather implicitly determined by the remaining investment horizon and the prevailing term structure, which in turn is determined by the current short rate level and the current CIR regime. Here, we consider a remaining investment period of 10 years.[34]

[33] See van Binsbergen and Brandt (2007) who investigate preventive and punitive aspects explicitly in an asset liability management framework.

[34] In practice, strategies like the Constant Proportion Portfolio Insurance (CPPI) that pursue short-term downside protection are also quite popular. In contrast to a long term risk management rule, they target at a revolving short term downside protection though the underlying investment horizon may be considerably longer. See Herold, Maurer, Stamos, and Vo (2007) for an analysis of short-term hedging effectiveness and long-term costs of these strategies.

[Table 4 about here.]

Similarly to the unconstrained case, intertemporal hedging demands require the dynamic investor to adjust her stock allocation compared to the myopic investor. The top subpanels indicates that the gain over a simpler myopic strategy is larger than in the unconstrained case but still less than one percent for nearly all states and risk aversions. Only a very risk averse investor with $\gamma = 10$ who faces a high volatility regime at low short rate levels can improve a little better. We find this result surprising, because once an investor hits the CVaR constraint, it is likely that she will be restricted for an extended period or even stopped out for the remaining investment period. Our asset allocation results also show that the dynamic investor tries to avoid such situations strategically by reducing her risky assets, even if the funding ratio is well above a critical level. Nevertheless, in a similar framework van Binsbergen and Brandt (2007) report, that a VaR constraint does not induce further hedging demands, but instead decreases the ability of the investor to act strategically. They argue that the VaR constraint did not result in a kinked utility function, but led only to a flat peak in the CRRA utility, which was common for most portfolio problems involving power utility. A flat surface around the optimum, however, implies that even large deviation from the optimal solution only result in small utility losses.

The costs for the money back guarantee are displayed in the bottom subpanel. They are derived by comparing the unconstrained dynamic to the constrained dynamic investment rule, hence both polices account for intertemporal hedging demands. In contrast to previous findings who report very low costs from money back guarantees, we can not conclude this in general. Compared to an optimal dynamic unconstrained solution, the loss due to risk management can be significant for unfavourable initial states. The costs increase with decreasing risk aversion as well as decreasing short rate levels, which is intuitive as less risk averse investors prefer riskier investment that are more likely to be affected by a CVaR constraint. Furthermore, at lower short rate levels the present value of liabilities are higher, which decreases the funding level and tightens the CVaR constraint. By contrast, a very risk averse investor is virtually not impacted by the CVaR constraint as she prefers a large portion of bonds anyway. Bonds are also a natural hedge against the interest rate risk in the liabilities. Whenever the remaining investment period is less that five years—which is the duration of the bond investment—the liability risk could be matched effectively by a duration hedge. Therefore, the costs for $\gamma = 5$ and $\gamma = 10$ investors are negligible. For low risk averse investors with $\gamma = 2$, the costs of money back guarantees are not deniable. With at average short rate levels, the costs are ranging between about 0.60 to 0.75 percent, and at low short rate levels, the costs are ranging about 1.25 to 1.95 percent.

6 Savings Plans and Money Back Guarantees

6.1 Asset Allocation Results

In this section, we extend our previous analysis to savings plans. In contrast to a single lump sum investment, the investor contributes to the investable wealth during the investment period. This reflects the savings and investment decision of an individual retirement investor more realistically. Over her life cycle, the individual investor typically starts with low financial but high human capital and allocates her current labour and financial income between saving and consuming. Hence, investment decisions are made over varying financial wealth with potential in- and outflows from the invested capital. In this paper, we focus on the money manager's investment objective and abstract from the individual's saving and consumption decision. To this end, we consider a simple yet realistic saving rule with deterministic absolute constant regular contributions.[35] More precisely, over a 10-year investment the investor allocates a constant absolute amount, say \$1, at the beginning of each period. The specific saving amount is irrelevant, because the CRRA utility is homothetic in wealth.

Essentially, a deterministic savings plan is equivalent to a lump sum investment with a partially restricted investment policy. A savings plan with T regular absolute constant contributions can alternatively be described as a portfolio of T zero coupon bonds with a maturity structure that matches the deterministic contribution dates. For instance, in the first period the investor is free to invest \$1, while she holds additional $T-1$ bonds with \$1 notional each. At each future contribution date one of the bonds expires and the investor can invest the repaid amount of \$1 freely. The last contribution is made at $T-1$. In the last period, the investor is unrestricted as in in the lump sum case. Each periodic savings amount w_t is invested according to the portfolio weights ω_t and the process for the overall contributed and invested amount W_t can be described as

$$W_{t+1} = \sum_{i=0}^{t} \prod_{j=i}^{t} w_i \left(\omega_j' R_{j+1}^e + R^f \right), \tag{44}$$

where R_{t+1}^e denotes the vector of excess returns and R^f the periodic risk free return. The process for the total value of the savings plan S_t is described by

$$S_{t+1} = W_{t+1} + w_{t+1} + \sum_{i=2}^{T-t-1} w_{t+i} P_{t+1}^{i-1} \quad \forall t < T-1. \tag{45}$$

S_{t+1} contains the accumulated wealth on the received savings contributions W_{t+1}, the next period's deterministic saving contribution w_{t+1}, and the next period's present value of future savings contributions. In $T-1$ we do not expect any further contributions, and hence $S_T = W_T$.

The portfolio decision affects only the process on the contributed amount W_{t+1}, because future contributions are not at the investor's disposal yet. Compared to an unconstrained lump sum investment as in section 4, the complexity of the optimisation problem increases for savings plan, because the current wealth level represents an additional state variable. At any point of time, we know the present value of the future savings contribution and the value of the contributions made so far, but the accumulated return made on the contributed capital is unknown. We need this information, however, in order to determine the fraction of our wealth that can be invested freely.

In the constrained case, a money back guarantee is written on each contributed dollar. For instance, in the first period a minimum of \$1 is guaranteed at the end of the 10-year

[35] Note, in our framework any other deterministic savings rule that may or may not be related to the current wealth level could be considered as well.

investment horizon, in the second period \$2, and in the last period \$10. In the case of money back guarantees, we restrict the investment policy analogous to the lump sum case. The next period's CVaR of the funding level has to be greater than unity. Thereby, the money back guarantee is active for all the contributions made so far. The risk constraint becomes

$$CVaR_{\alpha}\left(\frac{W_{t+1}+w_{t+1}}{\left(\sum_{i=0}^{t+1} w_i\right) e^{-y_{t+1}^{\tau-1}(\tau-1)}}\right) = CVaR_{\alpha}(C_{t+1}) \geq 1. \tag{46}$$

where C_{t+1} is the funding level defined over the next period's wealth W_{t+1} and the next period's saving rate w_{t+1}. Despite the fact that the savings contribution w_{t+1} is deterministic, its impact on the funding level is still random, because the next period's term structure is stochastic. Hence, we do not know how much risk budget the next savings contribution will add to our current risk budget. The lower the next period's present value of w_{t+1}, the more risk budget is added and vice versa. Note, the money back guarantee is less restrictive for a savings plan than for a lump sum investment, because future savings contribution always add additional risk budget. Savings contributions enter with their full amount in the numerator, but only with its present value in the denominator of the funding level. Hence, the funding level increases with each contribution ceteris paribus and the investor may never be truly stopped out as long as she awaits sufficient future savings contributions.

This, however, raises difficulties in our analysis, because whenever the investor faces a funding level below unity, she still can regain discretion about the portfolio policy at later periods. This is not the case for lump sum investments, because the current funding level would be locked-in by investing in the bond that fully matches the liability's maturity. Furthermore, if and when the investor regains discretion in a savings plan is stochastic and depends on the future liability process. This problem could be solved with an additional state variable, which would heavily increase computational time. Therefore, we apply an approximation for these cases. Whenever the investor cannot meet the CVaR constraint for the next period, she has to invest in the bond that matches the duration of her liability for the next period. For the evaluation of the value function, we truncate the funding level from below at unity. This is similar to restricting the investor to apply a constrained allocation with an assumed funding level at unity for later periods. Hence, our results may tend to underestimate the exact costs of money back guarantees for savings plan.

[Figure 3 about here.]

Figure 3 displays the optimal allocation to stocks for a savings plan with and without a money back guarantee. In both cases, the regime is set to the low volatility regime for the stock and the short rate process. For the unconstrained case in the left panels, we vary the short rate level, and for the constrained case in the right panels, we vary the funding level with the short rate set at its historical mean. The displayed stock fraction refers to the contributed and accumulated wealth so far. In the unconstrained case, an aggressive investor with $\gamma = 2$ is always fully invested in stocks, except at high short rate levels. Then, her allocation reduces to slightly above 80 percent for an one period investment. This is a more aggressive allocation to stocks than in the lump sum case, where at this short rate level the allocation is decreasing for longer investment periods. The latter is true for more risk averse investors, too. In contrast to the lump sum investment, the stock allocation for savings plans increases for longer investment periods. This result is known from previous work, especially in the life cycle household finance literature. Recall, a savings plan is equivalent to a lump sum investment with a large fraction bound to bond investment at the beginning. Hence, the investor allocates a larger fraction to stocks to compensate this effect, especially at the beginning of her investment period. As the savings plans matures, less capital is bound to bonds and the investor can shift to her overall desired allocation.

The constrained case shows non-monotonic allocation patterns, except for low risk aversion. An aggressive investor is fully invested in stocks at high funding levels, but as the funding level decreases, the CVaR restriction becomes more binding and the investor reduces her allocation to stocks monotonically with increasing time horizon. For higher risk aversions, the stock allocation increases first, but with increasing time horizon the duration of the liability increases as well. Both amplifies the funding level's volatility and enforces the CVaR restriction. Especially at low funding levels, the restriction to stocks can be severe; for all risk aversions the stock fraction is capped at about 20 percent for a 10 year investment, while the unconstrained fraction were 100 percent.

6.2 Welfare Analysis

[Table 5 about here.]

Table 5 displays the welfare costs of the money back guarantee for savings plans. Here, the costs are given as percentage of the savings plan's present value. Recall, each savings plan comprises ten annual payments, and hence the initial present value of each savings plan depends on the current state, which is given by the regime and the short rate. For better comparability, we normalize the costs by the initial present value of the savings plan.

The costs are the lowest for conservative investors, as they desire a larger bond portion. Because of the saving plan, the conservative investor holds implicitly a higher bond portion anyway. By contrast, the aggressive investor's wealth process is more correlated to the liabilities due to the higher implicit bond holdings. Because she cannot invest aggressively in early years, her risk increases to enter a constrained policy path, where she has to stick to low equity holdings. Starting in the low volatility short rate regime, the costs range from 0.38 to 1.09 percent, depending on the initial short rate. Costs are slightly lower in the high volatility short rate regime ranging from 0.34 to 0.83 percent. For $\gamma = 5$ costs do not exceed 0.36 percent in any case, and for $\gamma = 10$ costs are virtually zero.

7 Conclusion

In this paper we have analysed the impact of money back guarantees on optimal long-term asset allocations under regime switching stock returns and regime switching CIR-style term structures. The applied asset model is capable of generating a rich set of investment scenarios with conditional volatilities, non-normal asset returns, and non-linear time variation in the term structure. The CRRA investor maximises her utility over terminal wealth with respect to a CVaR constraint that monitors the probability of falling short the money back guarantee.

At each period, the liability is discounted by the current yield with maturity equal to the liability's duration. Whenever, the funding level is sufficiently high, the investor can invest freely, otherwise, she has to adopt her strategy to meet the CVaR constraint.

We present results for the optimal unconstrained policy as well as for the CVaR constrained policy, which are both derived in a dynamic framework in discrete time. Welfare costs for myopic lump sum investments are measured against the corresponding optimal dynamic solution; costs from the money back guarantee for lump sum investments and savings plans are derived from comparing against the optimal unconstrained policies. In line with previous research, we find that time variation in asset returns, does induce intertemporal hedging demands but potential gains compared to simpler myopic strategies are small. With respect to the financial literacy of average investors in practice, we doubt that any efforts towards implementing the dynamic strategy were justified. For a lump sum investment, the costs from risk management are negligible for very risk averse investors, because they naturally prefer bonds over stocks. Of course, bonds are less riskier than stocks and also provide an effective hedge against the liability risk. Less risk averse investors, however, suffer from such guarantees, especially in a low interest rate environment. Nevertheless, in practice most plans are designed as saving plans, in which the investors contribute a specific amount each period instead of investing all at the beginning. In that setting, costs from the money back guarantee are similarly small.

From our perspective, the results are very supportive for money back guarantees in individual retirement accounts, especially if implemented as in the German Retirement Savings Act. They provide retirees with a minimum level of benefit at very low economic costs. Conflicts arising from moral hazard, lack of financial literacy, and high upfront insurance costs are mitigated from this conditional solvency framework. In fact, if capital markets develop fairly well, no costs from the money back guarantee would ever be incurred as the investor could invest freely throughout. Hence, the term *conditional* solvency framework is most suitable, because risk capital needs only to be set aside by the money manager if necessary.

Further research in this direction could target more ambitious guarantees than a nominal minimum zero return. However, the risk management scheme considered here is not readily applicable for positive returns, because upfront costs can not be generally ruled out in this case. We also choose a simple fix maturity bond portfolio, and do not consider the impact of duration management explicitly. Hence, more complex liability structures could be introduced with the need for more emphasis on the bond portfolio management. Furthermore, risk management could be moved from the CVaR constraint into the objective functions directly by incorporating loss aversion. It would be interestingly to see if welfare losses under loss aversion utility prove to be substantially different to those in the CRRA case. And finally, additional important sources of risk could be considered as well. For instance, parameter and model uncertainty would not only impact the portfolio policy, but also the risk management, as conditional CVaR estimates depend on the uncertain distribution as well.

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Table 1

Parameter Estimates for the Two-Regime Asset Model

The table reports the pseudo-maximum likelihood estimates for the two-regime CIR-style term structure model and the excess stock return process on a monthly basis. Numerically derived standard errors are in parenthesis. The 3-month T-bill is used as proxy for the short rate, which is jointly estimated with the market price of risk λ by assuming pricing errors for the 6-month T-bill and the 5-year bond yield. Parameters are given on a monthly basis.

	Regime LV		Regime HV	
κ	0.0154	(0.0020)	0.0385	(0.0084)
θ	0.0023	(1.07E-04)	0.0071	(0.0014)
σ	0.0029	(1.41E-04)	0.0070	(0.0005)
λ	-0.0122	(0.0013)	-0.0089	0.0078
σ_{6m}	1.0E-04	(5.1E-06)	2.0E-04	(1.2E-05)
$\sigma_{6m,5y}$	7.3E-04	(3.7E-05)	8.9E-04	(5.8E-05)
$\rho_{6m,5y}$	0.4883	(0.0528)	0.6098	(0.0560)
Regime LV	0.9787	(0.0023)	0.0920	(0.0106)
Regime HV	0.0213		0.9080	
Long run	0.8122		0.1878	
μ	0.0056		0.0056	
v	0.0334	(0.0041)	0.0650	(0.0111)
Regime LV	0.9498	(0.0243)	0.1486	(0.2000)
Regime HV	0.0502		0.8514	
Long run	0.7488		0.2512	

Table 2

Optimal Allocation to Stocks - Unconstrained Policy

The table reports the optimal unconstrained dynamic allocation to stocks in percent for different investment horizons in years. The initial short rate level is set either at the historical mean of the 3-month T-bill, -1 standard deviation, or $+1$ standard deviation. The different panels (top to bottom) vary the initial regimes; the low volatility stock regime (SLV), the high volatility stock regime (SHV), the low volatility CIR regime (CLV), and the high volatility CIR regime (CHV). The columns correspond to varying risk aversions and initial short-rate levels. In each subpanel, the first row display the optimal one-period allocations, while the remaining rows display the intertemporal hedging demands.

	$\gamma = 2$			$\gamma = 5$			$\gamma = 10$		
Periods	$-1\sigma_x$	μ_x	$+1\sigma_x$	$-1\sigma_x$	μ_x	$+1\sigma_x$	$-1\sigma_x$	μ_x	$+1\sigma_x$
SLV-CLV									
1	1.000	1.000	0.834	0.625	0.497	0.370	0.347	0.281	0.217
2	0.000	0.000	−0.008	−0.013	−0.013	−0.015	−0.018	−0.017	−0.019
5	0.000	0.000	−0.023	−0.042	−0.042	−0.046	−0.057	−0.055	−0.058
10	0.000	0.000	−0.032	−0.059	−0.061	−0.066	−0.078	−0.077	−0.082
SHV-CLV									
1	1.000	0.839	0.631	0.442	0.357	0.274	0.244	0.200	0.158
2	0.000	−0.005	−0.005	−0.009	−0.009	−0.010	−0.013	−0.012	−0.012
5	0.000	−0.013	−0.015	−0.029	−0.028	−0.030	−0.039	−0.037	−0.038
10	0.000	−0.019	−0.021	−0.041	−0.040	−0.043	−0.053	−0.051	−0.053
SLV-CHV									
1	1.000	1.000	0.998	0.640	0.558	0.479	0.385	0.342	0.305
2	0.000	0.000	−0.009	−0.027	−0.028	−0.031	−0.034	−0.035	−0.039
5	0.000	0.000	−0.034	−0.089	−0.088	−0.095	−0.115	−0.112	−0.118
10	0.000	0.000	−0.055	−0.131	−0.129	−0.136	−0.165	−0.160	−0.168
SHV-CHV									
1	1.000	0.894	0.760	0.461	0.405	0.353	0.265	0.245	0.220
2	−0.001	−0.010	−0.011	−0.019	−0.019	−0.021	−0.014	−0.024	−0.026
5	−0.009	−0.029	−0.032	−0.062	−0.060	−0.064	−0.069	−0.076	−0.080
10	−0.020	−0.042	−0.045	−0.090	−0.087	−0.092	−0.103	−0.108	−0.113

Table 3

Welfare Costs of Myopia - Unconstrained Policy

The table reports the welfare cost of myopia in cents per dollar for alternative initial regimes, risk aversions, and short rate levels. The myopic investor always applies the optimal one-period allocation irrespective of the remaining investment horizon. The columns correspond to varying risk aversions and initial short rate levels. The initial short rate level is set either at the historical mean of the 3-month T-bill, -1 std. dev., or $+1$ std. dev. The alternative initial regimes comprise the low volatility stock regime (SLV), the high volatility stock regime (SHV), the low volatility CIR regime (CLV), and the high volatility CIR regime (CHV).

	$\gamma = 2$			$\gamma = 5$			$\gamma = 10$		
	$-1\sigma_x$	μ_x	$+1\sigma_x$	$-1\sigma_x$	μ_x	$+1\sigma_x$	$-1\sigma_x$	μ_x	$+1\sigma_x$
SLV-CLV	0.00	0.00	0.01	0.13	0.13	0.14	0.45	0.42	0.44
SHV-CLV	0.00	0.00	0.01	0.12	0.13	0.13	0.43	0.40	0.41
SLV-CHV	0.00	0.00	0.01	0.22	0.22	0.24	0.69	0.68	0.73
SHV-CHV	0.00	0.01	0.01	0.19	0.19	0.21	0.92	0.59	0.63

Table 4

Welfare Costs for Constrained Policies

The table reports the welfare costs of constrained myopic investing (top subpanels) and the money back guarantee (bottom subpanels) for CVaR confidence levels of 95 percent. The constrained myopic policy is compared against the constrained dynamic strategy, while the costs for the money back guarantee is derived by comparing the dynamic unconstrained policy versus the dynamic constrained policies. The different columns corresponds to different risk aversions and initial regimes: the low volatility stock regime (SLV), the high volatility stock regime (SHV), the low volatility CIR regime (CLV), and the high volatility CIR Regime (CHV). The initial short-rate level is set either at the historical mean of the 3-month T-bill, one standard deviation below the mean, or one standard deviation above the mean.

	$\gamma = 2$			$\gamma = 5$			$\gamma = 10$		
	$-1\sigma_x$	μ_x	$+1\sigma_x$	$-1\sigma_x$	μ_x	$+1\sigma_x$	$-1\sigma_x$	μ_x	$+1\sigma_x$
Myopia									
SLV-CLV	0.31	0.19	0.09	0.34	0.17	0.16	0.55	0.49	0.50
SHV-CLV	0.66	0.25	0.08	0.30	0.16	0.15	0.53	0.46	0.47
SLV-CHV	0.25	0.15	0.11	0.35	0.26	0.27	0.84	0.75	0.80
SHV-CHV	0.56	0.23	0.10	0.29	0.22	0.23	1.15	0.66	0.69
Money Back Guarantee									
SLV-CLV	1.79	0.76	0.22	0.09	0.01	0.00	0.00	0.00	0.00
SHV-CLV	1.93	0.69	0.21	0.07	0.01	0.00	0.00	0.00	0.00
SLV-CHV	1.24	0.60	0.21	0.05	0.01	0.00	0.00	0.00	0.00
SHV-CHV	1.34	0.59	0.20	0.05	0.01	0.00	0.00	0.00	0.00

Table 5

Welfare Costs of Money Back Guarantee - Savings Plan

The table reports the welfare costs of the Money Back Guarantee in cents per dollar calculated as certainty equivalence of the unconstrained versus the $\text{CVaR}_{0.95}$-constrained optimal dynamic savings plan. The columns correspond to varying risk aversions and initial short-rate levels. The initial short rate level is set either at the historical mean of the 3-month T-bill, -1 std. dev., or $+1$ std. dev. The alternative initial regimes comprise the low volatility stock regime (SLV), the high volatility stock regime (SHV), the low volatility CIR regime (CLV), and the high volatility CIR regime (CHV).

	$\gamma = 2$			$\gamma = 5$			$\gamma = 10$		
	$-1\sigma_x$	μ_x	$+1\sigma_x$	$-1\sigma_x$	μ_x	$+1\sigma_x$	$-1\sigma_x$	μ_x	$+1\sigma_x$
SLV-CLV	1.08	0.63	0.38	0.35	0.13	0.04	0.02	0.00	0.00
SHV-CLV	1.09	0.64	0.38	0.36	0.14	0.04	0.02	0.00	0.00
SLV-CHV	0.83	0.53	0.34	0.24	0.10	0.04	0.01	0.00	0.00
SHV-CHV	0.83	0.53	0.34	0.25	0.10	0.04	0.01	0.00	0.00

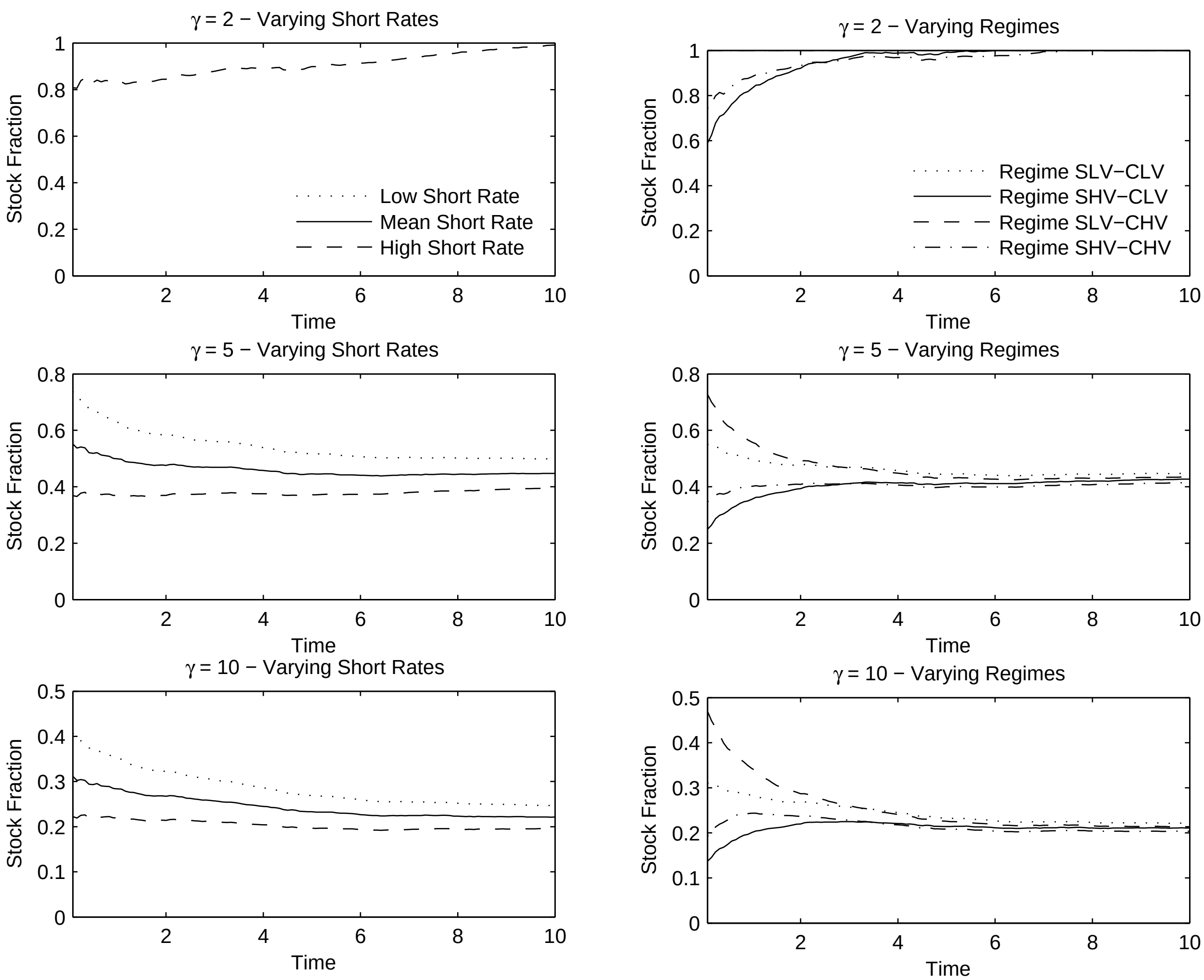

Figure 1: Optimal Stock Allocation for a Buy-and-Hold Investor.
The figure plots the optimal allocation to stocks for an unconstrained buy-and-hold investor against different investment horizons in years. The left panels fix the initial regime at the low volatility regime and vary the short rate level from low (-1 std. dev.), to mean (its historical average), and to high ($+1$ std. dev.). The right panels fix the short rate level at its historical mean and vary the regimes. The risk aversion increases from the top to the bottom panels.

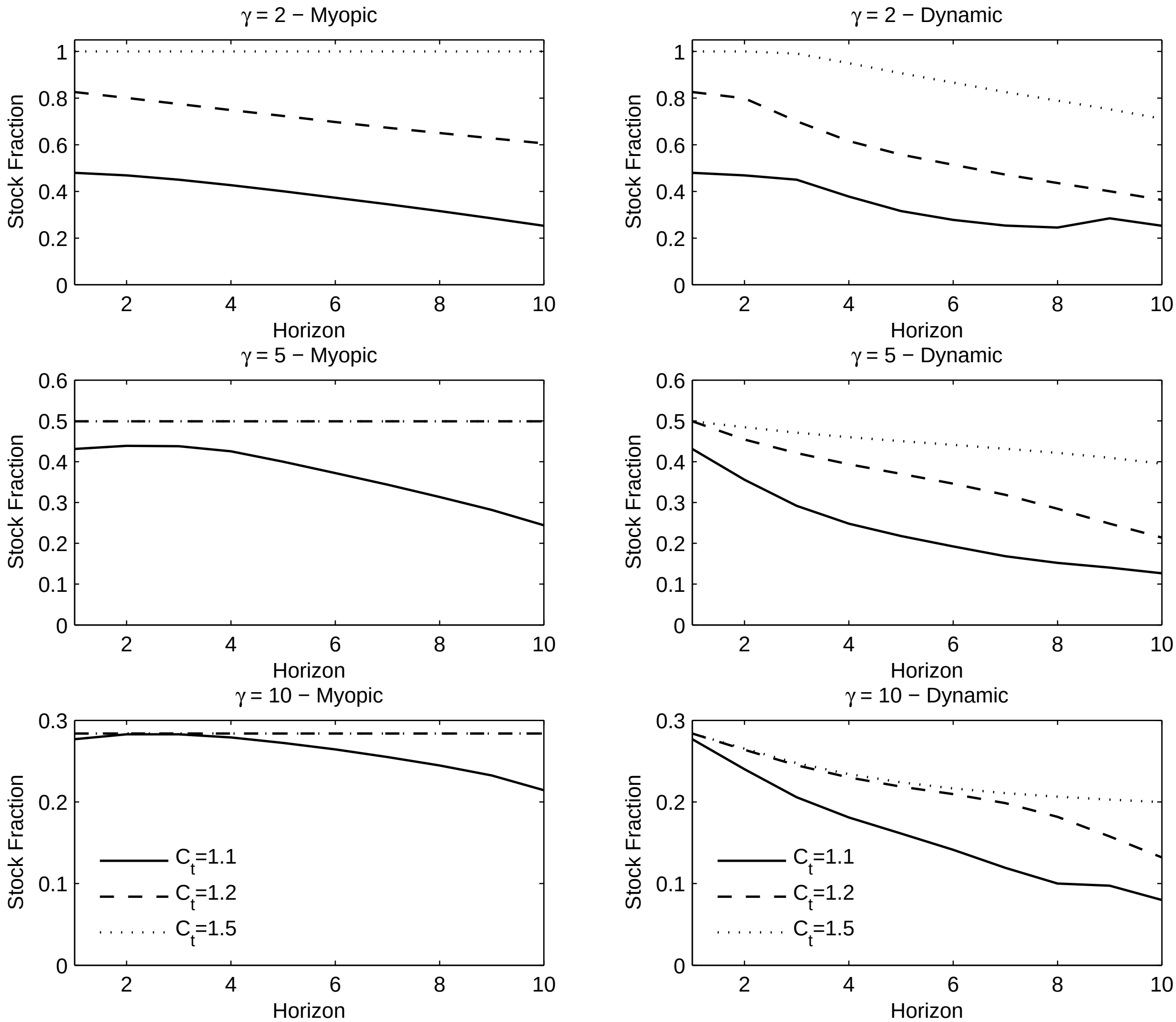

Figure 2: Dynamic and Myopic Stock Allocation for the Constrained Policy.
The figure plots the optimal allocation to stocks for a $\text{CVaR}_{0.95}$-constrained investor against different investment horizons in years. The stock and the CIR process prevail at the low volatility regime with the short rate set at its historical mean. The left panels display the dynamic solution, the left the myopic solution. The different lines correspond to different funding levels C_t as defined in equation (39). The risk aversion increases from top to bottom panels.

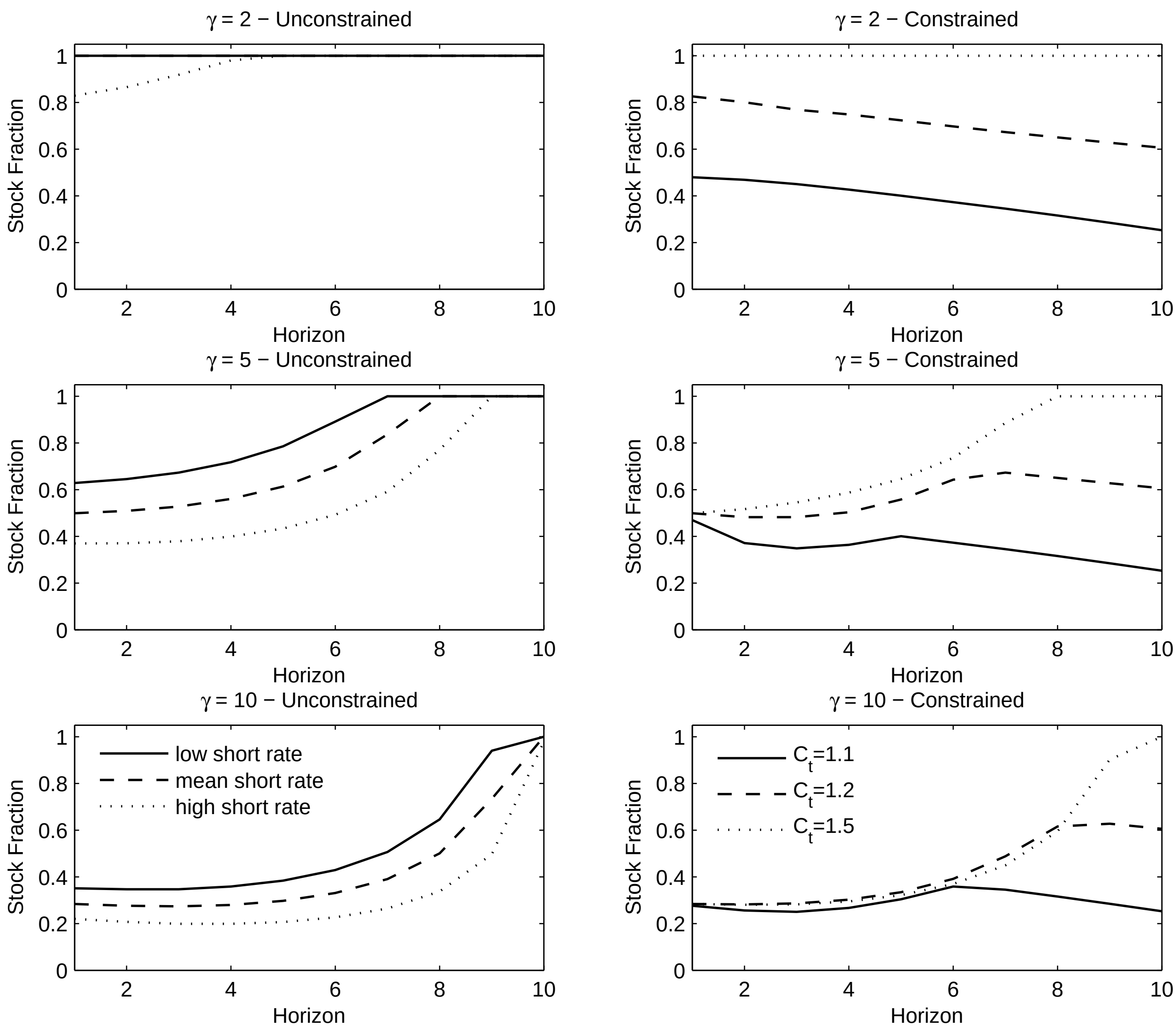

Figure 3: Stock Allocation for an Unconstrained and a Constrained Savings Plans.

The figure plots the optimal allocation to stocks against the investment horizon for an investor with constant yearly savings contributions. The left panels display the unconstrained case for varying short rate levels x_t. The right panels display the $\text{CVaR}_{0.95}$-constrained case for varying funding levels C_t as defined in equation (39) and the short rate level set at its historical mean. The risk aversion γ increases from top to bottom panels.