PENSION MATHEMATICS
with Numerical Illustrations

Second Edition

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Chapter 5

Pension Liability Measures

A variety of liability measures are associated with pension plans, each one having a specified purpose. Some liabilities represent the financial obligations of the plan, either on a plan termination or ongoing basis, while others simply represent mathematical byproducts of various actuarial cost methods used for funding pension plans. Although the latter are not liabilities in the true sense of the word, they are referred to as *actuarial liabilities* to distinguish them from the term liability as used in the fields of finance and accounting. The accounting profession has promulgated several specific pension liability measures, and another set of liabilities is defined by federal statutes in determining minimum required and maximum tax deductible contributions. Finally, since none of these liabilities may represent what management believes to be the "true" long-term financial obligation of the plan, *economic liabilities* are sometimes used to evaluate a plan's funded status.

The purpose of this chapter is to present the fundamental mathematics of alternative liability measures. The specific liabilities defined by various federal statutes are presented in Chapter 10, at which point the rules associated with pension funding are discussed. Similarly, the liabilities used in pension accounting are presented in Chapter 11, at which point the entire subject of pension accounting is presented. Economic liabilities are discussed in Chapter 14. Although the liabilities presented in this chapter encompass the general definitions of the various federal statutory, accounting, and economic liabilities, there are other aspects to these liabilities that are discussed in their respective chapters.

At this point only the liability associated with retirement is considered, and then only for a single age r. The corresponding liabilities for vested termination benefits, disability benefits, death benefits, and early retirement benefits are defined in Chapters 8 and 9.

PLAN TERMINATION LIABILITY

The plan termination liability (PTL), sometimes referred to as the plan's *legal liability*, is equal to the present value of all accrued benefits, both for active and retired employees. Assuming that the benefit is in the form of an annuity payable for the lifetime of the retiree, equation (5.1a) defines this liability for a participant age x prior to retirement, while (5.1b) is applicable after retirement:

$$(PTL)_x = B_x \; {}_{r-x}p_x^{(m)} \; v^{r-x} \; \ddot{a}_r, \qquad \text{for } x \leq r \qquad (5.1a)$$

where

B_x = accrued benefit as defined by the plan

${}_{r-x}p_x^{(m)}$ = probability of living from age x to age r

v^{r-x} = interest discount from age x to age r[1]

$\ddot{a}_r$ = present value, at age r, of a life annuity;

$$(PTL)_x = B_r \; \ddot{a}_x, \qquad \text{for } x \geq r \qquad (5.1b)$$

where

B_r = retirement benefit payable for life

$\ddot{a}_x$ = present value, at age x, of a life annuity.

The $(PTL)_x$ function increases sharply with age prior to retirement, since the first three factors in (5.1a) increase with age, while the fourth term is constant. After retirement, the $(PTL)_x$ function decreases according to the annuity function, since the benefit function is constant.

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[1] In practice, the interest rate used to evaluate the liability associated with a plan termination may be based on non-uniform rates instead of a constant rate, as assumed in this presentation.

The $(PTL)_x$ function for active employees utilizes the mortality survival function, whereas all of the other liability measures presented in this chapter involve the composite survival function, which includes decrements for termination and disability. The mortality survival function is appropriate for the $(PTL)_x$ function since only death would prevent the participant from receiving the accrued benefit at retirement if the plan were terminated.[2] The participant's future employment status or disability status would have no bearing on the receipt of the accrued retirement benefit.

The mathematical definition of $(PTL)_x$ for retired participants is the same for all liability measures. Hence, the remainder of this chapter is devoted to alternative liability measures for active participants (i.e., all x's $\leq r$).

PLAN CONTINUATION LIABILITY

The plan continuation liability for accrued benefits, sometimes referred to as the *ongoing liability* for accrued benefits, measures the financial obligation under the assumption that the plan will continue to exist. In this case, future employment and disability statuses are relevant. Equation (5.2a) defines this liability measure for an active participant at age x:

$$^{AB\,r}(PCL)_x = B_x \; {}_{r-x}p_x^{(T)} \, v^{r-x} \, \ddot{a}_r . \tag{5.2a}$$

The AB prescript to the plan continuation liability symbol indicates that the liability is based on the accrued benefit as defined by the plan. The first edition of this book expressed this liability in terms of a service prorated projected benefit, defined by (3.15b). Other experts believe it should be expressed in terms of a salary prorated benefit, as defined by (3.16b), or the "accrued benefit" implicitly defined by a given actuarial cost method. Any of these definitions can represent a meaningful plan continuation liability, depending on one's philosophy. The plan continuation liability also includes ancillary benefits; hence, the prescript r is used to denote that only retirement benefits are being evaluated at this point, a convention used hereafter until the liability associated with ancillary benefits is defined.

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[2]Under federal law, even non-vested employees become fully vested in their accrued benefits if a single-employer plan is terminated with sufficient assets.

The plan continuation liability and the plan termination liability can be expressed in terms of each other:

$$ {}^{AB\,r}(PCL)_x = \frac{{}_{r-x}p_x^{(T)}}{{}_{r-x}p_x^{(m)}}\ {}^{r}(PTL)_x \tag{5.2b} $$

$$ = \frac{{}_{r-x}p_x^{(m)}}{{}_{r-x}p_x^{(T)}}\ {}^{r}(PCL)_x . \tag{5.2c} $$

Under identical actuarial assumptions, the ${}^{AB\,r}(PCL)_x$ function will be lower in value than the $(PTL)_x$ function until retirement, at which point they become equal. If the liability associated with termination, disability, and death benefits were to be included, however, the ${}^{AB\,r}(PCL)_x$ may equal or exceed the $(PTL)_x$. If the plan had no disability or death benefits and the employee were fully vested, then the two liability values would be equal if the vested termination liability were included in the ${}^{AB\,r}(PCL)_x$ function. In other words, in both cases only death prior to retirement would prevent the participant from receiving the accrued benefit.

As a practical matter, the actuarial assumptions used in evaluating these two liabilities are likely to be different. The interest rate used with the $(PTL)_x$ function, for example, might logically approximate the rate at which the plan sponsor could "sell" the liability to an insurance carrier, whereas the interest rate used for ${}^{AB\,r}(PCL)_x$ is likely to represent the plan sponsor's expected long-run return on plan assets. Thus, it is difficult to predict the relative values of the plan termination and plan continuation liabilities.

Table 5–1 shows these two liability values during the career of an age-30 entrant, expressed as a percentage of the age-65 value. Post-65 values are also provided through age 100. The model assumptions are used in the illustration, but without ancillary benefits included in ${}^{AB\,r}(PCL)_x$. Figure 5–1 plots the values in Table 5–1, graphically illustrating that the bulk of an employee's liability is associated with ages near retirement.

ACTUARIAL LIABILITIES

Several actuarial cost methods are used with pension plans, and each method has an associated actuarial liability. In general

TABLE 5–1

Plan Termination and Plan Continuation Liabilities as a Percentage of the Age-65 Value

x	$(PTL)_x$	$^r(PCL)_x$	x	$B_r\,\ddot{a}_x$
30	0.00	0.00	65	100.00
32	0.04	0.01	66	97.38
34	0.11	0.03	68	92.06
36	0.22	0.08	70	86.72
38	0.41	0.16	72	81.48
40	0.69	0.32	74	76.28
42	1.13	0.57	76	70.93
44	1.79	1.00	78	65.57
46	2.77	1.69	80	60.47
48	4.21	2.79	82	55.76
50	6.30	4.55	84	51.42
52	9.32	7.32	86	47.42
54	13.65	11.64	88	43.69
56	19.83	17.71	90	40.17
58	28.61	25.84	92	36.84
60	41.05	37.64	94	33.57
62	58.69	55.03	96	30.38
64	83.74	81.48	98	27.29
65	100.00	100.00	100	24.26

terms, a cost method's actuarial liability is equal to the present value of benefits allocated to date, which can be expressed as follows:

$$^r(AL)_x = B_x^{'}\ {}_{r-x}p_x^{(T)}\ v^{r-x}\ \ddot{a}_r, \tag{5.3}$$

where $B_x^{'}$ represents the benefits allocated under a given actuarial cost method, as discussed at a later point in this chapter. Observe that, if the benefit function is equal to the accrued benefit as defined by the plan, the expression is identical to the plan continuation liability given by (5.2a).

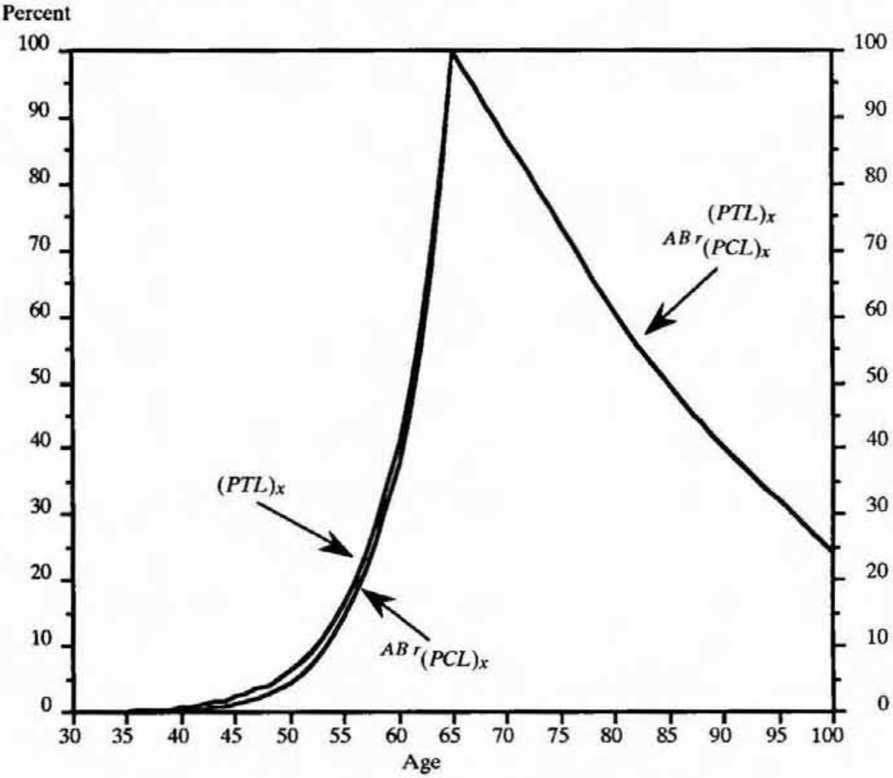
The actuarial liability of a given cost method may also be viewed as the portion of the participant's *present value of future benefits* (PVFB) allocated under the method. The $^r(PVFB)_x$ function equals the present value of the participant's *total projected* retirement benefit:

$$^r(PVFB)_x = B_r\ {}_{r-x}p_x^{(T)}\ v^{r-x}\ \ddot{a}_r. \tag{5.4a}$$

The $^r(PVFB)_x$ function is the same as the $^r(AL)_x$ function

FIGURE 5–1

Plan Termination and Plan Continuation Liabilities as a Percentage of the Age-65 Value



evaluated with B_r instead of B_x'. Since the actuarial liability represents the proportion of $^r(PVFB)_x$ allocated by the actuarial cost method being used, a generalized actuarial liability definition can be expressed in the following manner:

$$^r(AL)_x = k\ ^r(PVFB)_x, \tag{5.4b}$$

where k is a fraction dependent on each cost method and defined in subsequent sections of this chapter.

Accrued Benefit Method

The actuarial liability under the accrued benefit (AB) method, sometimes referred to as the *unit credit method*, is equal to the present value of accrued benefits (i.e., equation (5.3) evaluated with the accrued benefit as defined by the plan):

$$^{AB}\,{}^{r}(AL)_x = B_x \;{}_{r-x}p_x^{(T)}\; v^{r-x}\; \ddot{a}_r \tag{5.5a}$$

$$= \frac{B_x}{B_r}\;{}^{r}(PVFB)_x . \tag{5.5b}$$

In this case, the value of k in (5.4b) is a ratio of the accrued benefit to the age-r benefit. Note that the accrued benefit actuarial liability is defined mathematically to be the same as the $^{AB}\,{}^{r}(PCL)_x$ function; however, these values may not be equal for a given plan because different actuarial assumptions might logically be used for each liability measure.

Benefit Prorate Methods

There are two benefit prorate methods, generally referred to as *projected unit credit methods*. The actuarial liability under the first version uses the accrued benefit function defined by (3.15b), i.e., a service proration of the participant's projected retirement benefit yielding a constant dollar benefit allocated to each attained age. The actuarial liability is defined by equations (5.6a) and (5.6b), where the BD prescript denotes the "**b**enefit prorate, constant **d**ollar" version:

$$^{BD}\,{}^{r}(AL)_x = \frac{x-y}{r-y}\, B_r \;{}_{r-x}p_x^{(T)}\; v^{r-x}\; \ddot{a}_r \tag{5.6a}$$

$$= \frac{x-y}{r-y}\;{}^{r}(PVFB)_x . \tag{5.6b}$$

In this case, the value of k in (5.4b) is the ratio of current service to projected service at retirement.

The actuarial liability under the second version uses the accrued benefit defined by (3.15b), i.e., a salary proration of the participant's projected retirement benefit yielding a benefit allocated to each attained age equal to a constant percent of salary:

$$^{BP}\,{}^{r}(AL)_x = \frac{S_x}{S_r} B_r \;{}_{r-x}p_x^{(T)}\; v^{r-x}\; \ddot{a}_r \tag{5.7a}$$

$$= \frac{S_x}{S_r}\;{}^{r}(PVFB)_x . \tag{5.7b}$$

The BP prescript denotes the "**b**enefit prorate, constant **p**ercent" version. The value of k in (5.4b) is the ratio of the participant's cumulative salary at age x to the projected cumulative salary at age r.

These two liability measures will be somewhat greater than the accrued benefit actuarial liability discussed above, assuming the same actuarial assumptions.

Cost Prorate Methods

There are two cost prorate methods, sometimes referred to as *projected benefit cost methods* or *entry age cost methods.* Again the liabilities can be defined in terms of prorated retirement benefits, but in this case the proration is based on temporary employment-based annuities. As we will see in Chapter 6, the pension cost under one version is equal to a constant dollar amount throughout the employee's career, whereas the other version has costs equal to a constant percent of the employee's salary. The prescripts CD (cost prorate, constant dollar) and CP (cost prorate, constant percent) are used to designate each type.

The actuarial liability under the constant dollar version is given by (5.8a) and (5.8b), while the constant percent version is given by (5.9a) and (5.9b):

$$ {}^{CD\,r}(AL)_x = \frac{\ddot{a}^T_{y:\overline{x-y}|}}{\ddot{a}^T_{y:\overline{r-y}|}} B_r \; {}_{r-x}p_x^{(T)} \, v^{r-x} \, \ddot{a}_r \tag{5.8a} $$

$$ = \frac{\ddot{a}^T_{y:\overline{x-y}|}}{\ddot{a}^T_{y:\overline{r-y}|}} \, {}^r(PVFB)_x ; \tag{5.8b} $$

$$ {}^{CP\,r}(AL)_x = \frac{{}^s\ddot{a}^T_{y:\overline{x-y}|}}{{}^s\ddot{a}^T_{y:\overline{r-y}|}} B_r \; {}_{r-x}p_x^{(T)} \, v^{r-x} \, \ddot{a}_r \tag{5.9a} $$

$$ = \frac{{}^s\ddot{a}^T_{y:\overline{x-y}|}}{{}^s\ddot{a}^T_{y:\overline{r-y}|}} \, {}^r(PVFB)_x . \tag{5.9b} $$

In each case, the projected benefit is prorated by a ratio of temporary employment-based annuities, with the ratio increasing to unity by retirement.[3] The annuities were defined previously by (3.22) and (3.23), and the ratios represent the value of k in (5.4b) for each method.

If salary is a non-decreasing function of x, the following inequalities hold, indicating the relative size of the actuarial liabilities under each of the five cost methods described above:

$$0 \leq \frac{B_x}{B_r} \leq \frac{S_x}{S_r} \leq \frac{x-y}{r-y} \leq \frac{{}^s\ddot{a}^T_{y:\overline{x-y}|}}{{}^s\ddot{a}^T_{y:\overline{r-y}|}} \leq \frac{\ddot{a}^T_{y:\overline{x-y}|}}{\ddot{a}^T_{y:\overline{r-y}|}} \leq 1\,. \quad (5.10)$$

In theory there need not be any restriction on the value of k in (5.4b), provided that it is zero at age y and attains unity at age r. Consequently, an infinite number of actuarial liabilities exist, only a few of which are formally recognized.

Table 5–2 shows the various actuarial liabilities for an age-30 entrant under the model actuarial assumptions, all expressed as a percent of the age-65 values (values that is identical among the methods). Note that the actuarial liabilities are relatively small throughout most of the employee's career as compared to the age-65 value. These values indicate that a substantial portion of a plan's actuarial liability may be associated with individuals in or near retirement, even though the number of such individuals may be relatively small. Figure 5–2 shows a graph of the attained age pattern of the actuarial liabilities during an employee's career.

Figure 5–3 shows the various actuarial liabilities for the model pension plan, using the population in the 25th year of the simulation shown in Table 4–7. The values are all expressed as a percent of the present value of future benefits. These liabilities are determined for the entire plan membership, and they include the liabilities associated with ancillary benefits, a subject yet to be discussed. The liability associated with retired employees is constant among the various methods; however, the liability for active employees differs substantially. Since cost methods are designed to have plan assets accumulate to their respective actuarial liabilities, it is clear that different cost methods can have a significant effect on the ultimate level of plan assets. This, in turn, suggests that plan contributions during the early stages of

[3] At this point it is premature to provide the logic as to why these two actuarial liabilities can be defined by prorating the projected benefit by the annuity ratios, a subject taken up in Chapter 6.

funding will be higher for those methods with larger actuarial liabilities; however, such contributions eventually will be lower because of the investment returns associated with a larger asset base. These relationships will be explored more fully in later chapters.

TABLE 5–2

Actuarial Liability and $^r(PVFB)_x$ Functions as a Percentage of the Age-65 Value

	Accrued	Benefit Prorate Methods		Cost Prorate Methods		
	Benefit	Constant	Constant	Constant	Constant	
Age	Method	Percent	Dollar	Percent	Dollar	$^r(PVFB)_x$
30	0.00	0.00	0.00	0.00	0.00	0.95
31	0.00	0.01	0.04	0.11	0.20	1.24
32	0.01	0.02	0.09	0.25	0.45	1.55
33	0.02	0.04	0.16	0.43	0.75	1.90
34	0.03	0.07	0.26	0.64	1.08	2.27
35	0.05	0.11	0.38	0.89	1.46	2.68
36	0.08	0.16	0.53	1.18	1.88	3.12
37	0.12	0.22	0.72	1.52	2.36	3.61
38	0.16	0.30	0.95	1.92	2.90	4.16
39	0.23	0.41	1.23	2.39	3.51	4.78
40	0.32	0.54	1.56	2.93	4.19	5.47
41	0.43	0.71	1.96	3.55	4.96	6.24
42	0.57	0.92	2.44	4.27	5.83	7.10
43	0.76	1.19	3.00	5.10	6.80	8.07
44	1.00	1.52	3.67	6.04	7.90	9.16
45	1.30	1.93	4.45	7.13	9.13	10.39
46	1.69	2.43	5.38	8.37	10.52	11.77
47	2.17	3.05	6.47	9.80	12.09	13.32
48	2.79	3.82	7.76	11.43	13.86	15.08
49	3.57	4.77	9.27	13.30	15.86	17.07
50	4.55	5.94	11.04	15.45	18.14	19.32
51	5.78	7.37	13.13	17.90	20.71	21.88
52	7.32	9.13	15.58	20.72	23.64	24.78
53	9.24	11.29	18.46	23.95	26.96	28.08
54	11.64	13.93	21.83	27.65	30.74	31.83
55	14.63	17.16	25.77	31.87	35.02	36.08
56	17.71	20.37	29.34	35.44	38.51	39.50
57	21.40	24.18	33.40	39.42	42.38	43.30
58	25.84	28.68	38.01	43.88	46.69	47.52
59	31.18	34.03	43.28	48.90	51.50	52.24
60	37.64	40.41	49.34	54.57	56.92	57.56
61	45.46	48.07	56.35	61.04	63.08	63.62
62	55.03	57.32	64.54	68.50	70.16	70.59
63	66.81	68.60	74.22	77.22	78.43	78.72
64	81.48	82.54	85.84	87.55	88.21	88.36
65	100.00	100.00	100.00	100.00	100.00	100.00

FIGURE 5–2
Actuarial Liabilities as a Percentage of the Age-65 Values

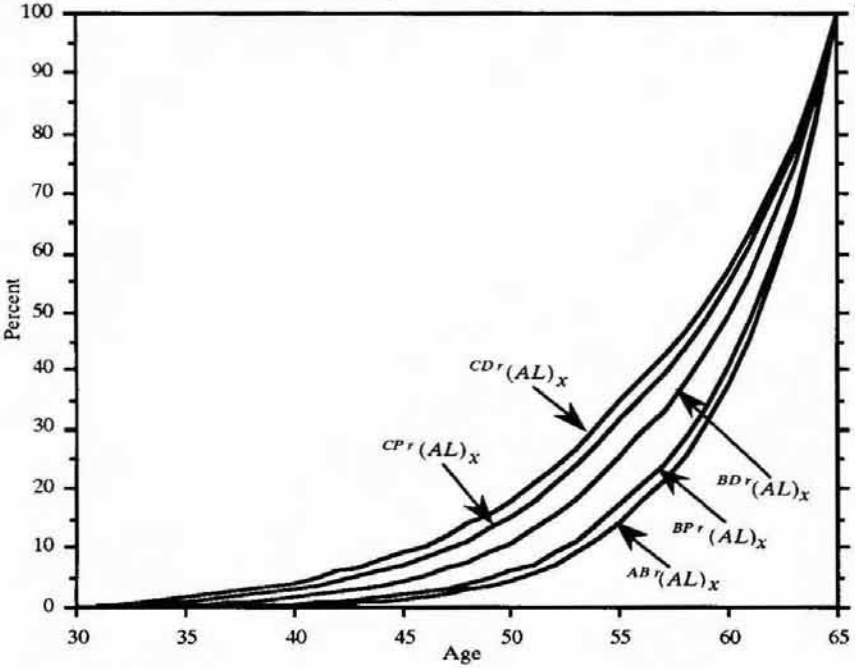


FIGURE 5–3
Actuarial Liabilities as a Percent of Present Value of Future Benefits Liability