PENSION MATHEMATICS
with Numerical Illustrations

Second Edition

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Chapter 4

Pension Plan Population Theory

BASIC CONCEPTS

This chapter deals with the population of pension plan members, which consists of several subpopulations. Active employees, of course, make up the primary group within the plan's membership, and the following discussion focuses on them. Retired employees represent another subpopulation, and for present purposes, this group is assumed to consist only of members who retired directly from the service of the employer. This is in contrast to defining a retired member as any one of several types of benefit recipients. Vested terminated employees make up a third group, which can be further divided into those in the benefit deferral period and those receiving benefits. Disabled employees make up the fourth subpopulation for plans providing disability benefits, and beneficiaries, generally surviving spouses, make up a fifth subpopulation.

Stationary Population

The discussion of pension plan populations begins at the most elementary level, namely, with the concept of a stationary population. A population is considered to be stationary when its size and age distribution remain constant year after year. If the decrement rates associated with the population are constant, and if a constant number of new entrants flows into the population each year, a stationary condition will exist after n years, where n equals the oldest age in the population less the youngest age.[1]

Understanding the concept of a stationary population can be facilitated by considering a simplified example. Assume that a population has four ages (x, $x+1$, $x+2$, and $x+3$), that the rates of decrement for each age are 1/4, 1/3, 1/2, and 1, respectively, and that 100 new employees are hired each year, all at age x. The first few years experience for a population exposed to these conditions is given in Table 4–1.

TABLE 4–1
Development of a Stationary Population

Year	Ages: / Decrement Rates:	x 1/4	$x+1$ 1/3	$x+2$ 1/2	$x+3$ 1	$x+4$	Total Size
1	*New Entrants* →	100					100
2	*New Entrants* →	100	75				175
3	*New Entrants* →	100	75	50			225
4	*New Entrants* →	100	75	50	25		250
5	*New Entrants* →	100	75	50	25	0	250
6	*New Entrants* →	100	75	50	25	0	250
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Stationary Age Distribution:		40%	30%	20%	10%	0%	100%

After the first year, the original group of 100 employees hired at age x are age $x+1$ and total 75 in number. Since 100 new employees are hired at age x each year, the total population at the beginning of year 2 is 175 members. Continuing this process for three years (i.e., until the beginning of year 4), produces a population with a constant size of 250 members and a stationary distribution, as shown in percentage form at the bottom of Table 4–1. Thus, the population becomes stationary after n years,

[1] If the population is assumed to be continuous over each age interval, rather than discrete as is assumed in this chapter for simplicity, then n would be equal to the first age at which no survivors exist less the youngest age. In other words, one year would be added to the value of n for the continuous case.

where n in this case is equal to three (i.e., the oldest age in the population, $x+3$, less the youngest age, x).

Since pension benefits are tied to service, it is also relevant to point out that the service distribution of a stationary population also becomes constant after n years.[2] Table 4–1 shows that 40 percent of the stationary population has zero years of service, 30 percent has one year of service, and so forth.

A pension plan population, unlike the example shown in Table 4–1, has multiple entry ages, and it is logical to inquire whether or not the concept of a stationary population applies in this case. To show that it does apply, one need only conceptualize a multiple entry age population as a series of single entry age populations, with each subpopulation representing a given entry age. Consequently, a multiple entry age population of active employees will become stationary after m years, where m equals the largest retirement-age/entry-age spread among the various subpopulations.

Mature Population

The concept of a mature population is only slightly different from, and somewhat more general than, a stationary population. In fact, a stationary population is a special case of a mature population. Both concepts involve a constant year-to-year age and service distribution, but whereas the stationary population maintains a constant size, this need not be the case for a mature population. If the increments to the population (newly hired employees) increase at a *constant rate*, the population will attain a constant percentage age and service distribution in precisely the same length of time as required for a population to become stationary. Moreover, the size of the mature population will grow at precisely the same *rate* as the growth in new entrants.

These characteristics are illustrated in Table 4–2 where the number of new entrants is doubled each year (i.e., a 100 percent growth rate). The decrement assumptions are the same as those used in Table 4–1. The age distribution, which is shown at the bottom of the table, is constant year after year, but considerably

[2] This assumes, of course, that the entry age distribution of new entrants is fixed.

different from the age distribution developed in Table 4–1. This is the case for the service distribution also.

TABLE 4–2
Development of a Mature Population

Years	*Ages:* / *Decrement Rates:*	x 1/4	$x+1$ 1/3	$x+2$ 1/2	$x+3$ 1	$x+4$	*Total Size*
1		100					100
2		200	75				275
3		400	150	50			600
4		800	300	100	25		1,225
5		1,600	600	200	50	0	2,450
6		3,200	1,200	400	100	0	4,900
7		6,400	2,400	800	200	0	9,800
8		12,800	4,800	1,600	400	0	19,600
.		.	.	.	.	.	.
.		.	.	.	.	.	.
.		.	.	.	.	.	.
Mature Age Distribution:		65%	25%	8%	2%	0%	100%

Throughout the remainder of this book, the term mature population will be used, even in those cases where a stationary population applies, since it is the more general of the two concepts.

Undermature and Overmature Populations

A population is considered to be undermature if its age and service distribution has a larger proportion of younger, short-service employees than that of a mature population that faces the same decremental factors, is of the same size, and experiences the same entry age distribution. An overmature population is one that has a disproportionately large number of employees at older ages and with longer periods of service than that of a mature population based on the same decrement and entry age assumptions. Generally, growing industries are characterized by firms having undermature populations, while declining industries have firms with overmature populations.

An example of an undermature population is given in Table 4–3, where the number of new entrants increases by 100 employees each year, representing a continually decreasing rate of growth. The membership distribution in this example, parenthetically expressed in percentage form, asymptotically approaches the same distribution as that for the stationary population discussed previously in Table 4–1. After 100 years the population's age and service distribution is nearly identical to that of the corresponding stationary population.

TABLE 4–3
Development of an Undermature Population

Years	*Ages:* / *Decrement Rates:*	x / 1/4	$x+1$ / 1/3	$x+2$ / 1/2	$x+3$ / 1	$x+4$	*Total Size*
1		100 (100%)					100
2		200 (73%)	75 (27%)				275
3		300 (60%)	150 (30%)	50 (10%)			500
4		400 (53%)	225 (30%)	100 (13%)	25 (3%)		750
5		500 (50%)	300 (30%)	150 (15%)	50 (5%)	0	1,000
6		600 (48%)	375 (30%)	200 (16%)	75 (6%)	0	1,250
7		700 (47%)	450 (30%)	250 (17%)	100 (7%)	0	1,500
9		800 (46%)	525 (30%)	300 (17%)	125 (7%)	0	1,750
.		.	.	.	.	.	.
.		.	.	.	.	.	.
100		10,000 (40.4%)	7,425 (30.0%)	4,900 (19.8%)	2,425 (9.8%)	0	24,750
.		.	.	.	.	.	.
.		.	.	.	.	.	.
.		.	.	.	.	.	.
Asymptotic Age Distribution:		40%	30%	20%	10%	0%	100%

Finally, Table 4–4 illustrates the development of an overmature population, created by hiring 1,000 employees in the first year and 100 *fewer* employees each year thereafter. The degree to which the population is overmature is determinable by comparing its age and service distribution to that of the stationary population based on the same decrement rates (given in the lower part of Table 4–4 for convenience).

TABLE 4–4
Development of an Overmature Population

Years	Ages: Decrement Rates:	x 1/4	$x+1$ 1/3	$x+2$ 1/2	$x+3$ 1	$x+4$	Total Size
1		1,000 (100%)					1,000
2		900 (55%)	750 (45%)				1,650
3		800 (41%)	675 (34%)	500 (25%)			1,975
4		700 (35%)	600 (30%)	450 (23%)	250 (13%)		2,000
5		600 (34%)	525 (30%)	400 (23%)	225 (13%)	0	1,750
6		500 (33%)	450 (30%)	350 (23%)	200 (13%)	0	1,500
7		400 (32%)	375 (30%)	300 (24%)	175 (14%)	0	1,250
9		300 (30%)	300 (30%)	250 (25%)	150 (15%)	0	1,000
Stationary Age Distribution:		40%	30%	20%	10%	0%	100%

Size-Constrained Population

For purposes of illustration, the populations discussed thus far have had their membership size as a dependent variable and the number of newly hired employees as the independent variable. This is not an appropriate assumption in dealing with pension plan populations, since the sponsoring firm determines the size of its employee group on business considerations. Switching the dependent and independent variables has significant implications for the resulting age and service distributions.

The data in Table 4–5 show the results of a time-dependent plan population based on the same assumptions as before except for membership size, which is assumed to be constant at 1,000 employees from the inception of the population onward. Although the data are presented in terms of numbers of employees, one need only shift the decimal point one position to the left to obtain the percentage distribution.

TABLE 4–5
Development of a Size-Constrained Population

Years	Ages: / Decrement Rates:	*x* 1/4	*x* + 1 1/3	*x* + 2 1/2	*x* + 3 1	*x* + 4	Total Size
1		1,000 *					1,000
2		250	750 *				1,000
3		313	188	500 *			1,000
4		391	234	125	250 *		1,000
5		488 *	293	156	63	0	1,000
6		360	366*	195	78	0	1,000
7		388	270	244 *	98	0	1,000
8		407	291	180	122*	0	1,000
9		411 *	305	194	90	0	1,000
10		391	308 *	203	97	0	1,000
11		399	294	205 *	102	0	1,000
12		402	299	196	103 *	0	1,000
13		401 *	302	200	98	0	1,000
14		398	301 *	201	100	0	1,000
15		400	299	200 *	101	0	1,000
16		400	300	199	100*	0	1,000
.		.	.	.	.	.	.
.		.	.	.	.	.	.
.		.	.	.	.	.	.
	Ultimate:	400	300	200	100	0	1,000

The age and service distribution of the constant-size population is quite erratic at first, but becomes less so as time progresses. Although the age distribution in this example converges to that of the stationary population discussed in Table 4–1, the convergence is not smooth. In fact, a continually smaller ripple is seen to flow through the population during each successive year. This ripple, which begins with the initial group of 1,000 employees hired, is noted by asterisks in Table 4–5.

A size-constrained population will *generally* converge to its stationary counterpart created without a size constraint. The length of time required for the convergence is a function of the number of ages in the population and the rates of decrements at each of its ages. Naturally, the more attained ages, other things being equal, the longer it will take for the size-constrained population to come within a predetermined tolerance level of its stationary counterpart. Moreover, the lower the age-specific rates of decrement, the longer it generally takes for a predetermined tolerance level to be reached. The latter generalization can be appreciated by considering the following extreme example.

Suppose the age-specific decrements are zero for each age up to the last age, at which point the decrement rate is 100 percent. If a size constraint is imposed, the population will never converge to the uniform distribution that would result in the absence of the size constraint. In this case the population's age distribution will cycle indefinitely, with each n-year cycle consisting of n different distributions of 100 percent of the membership at each possible attained age, where n is the number of ages in the population. At the other extreme is the case where the rate of decrement is 100 percent at each age. The population under this assumption will be mature from its inception, since no employees will survive beyond the first age, and the entire labor force will be rehired at this age each year.

Up to this point the discussion of a size-constrained population has dealt with a single entry age population. This is unrealistic for pension plans, since they conform to the more general multiple entry age, size-constrained population. When the multiple entry age situation was discussed in the context of a stationary population without a size constraint, it was suggested that one conceptualize the population as consisting of a collection of single entry age populations. The situation is not nearly as simple when a size constraint is imposed. For example, it does not follow that each entry age subsector of the overall population will receive the same number of new entrants as the number of employees decrementing from that subsector each year. This is the case even if the hiring age distribution is held constant over time, unless the population is mature. The ultimate effect is that the population tends to approach a mature status sooner than it takes the longest entry age subpopulation to become mature under a size constraint. A numerical illustration of the convergence pattern of a multiple entry age population is given in the following section.

MODEL PLAN POPULATION

Pension costs are analyzed both for individual participants and for various model pension plan populations in this book. When dealing with the plan as a whole, it is necessary to assume an age and service distribution of plan members, a specific salary structure of active employees, and a benefit structure of nonactive plan members. In the interest of generality, numerous plan populations, in varying states of maturity, are assumed for the numerical illustrations. Rather than select the model population arbitrarily, the experience of a single plan population is simulated over a period of 50 years. In order to simulate the various maturity statuses, the initial plan population is undermature and is then forced through a mature state to an overmature status. This is accomplished by first having the size of the population increase at a decreasing rate and eventually decrease at an increasing rate.

The hiring age distribution and salary scale for new entrants during the 50-year simulation is given in Table 4–6. The average hiring age is 28, and the salary scale reflects one half of the previously discussed merit scale from age 20 to each of the specific entry ages. The total salary of active employees throughout the 50-year simulation increases according to the productivity and inflation rates specified earlier, that is, 1 percent and 4 percent,

TABLE 4–6

Hiring Age Distribution and Salary Scale

Entry Age	*Hiring Distribution*	*Salary Scale*
20	0.277	1.0000
25	0.290	1.1171
30	0.152	1.2437
35	0.101	1.3747
40	0.086	1.5042
45	0.049	1.6252
50	0.016	1.7301
55	0.015	1.8122
60	0.014	1.8655

respectively. The annual rate of increase in salary for the total plan population, however, will be somewhat more than 5 percent because of the maturation that takes place in the population of active employees.

Table 4–7 shows various statistics for the simulated plan population. The number of active employees, expressed as a percentage of the initial group, doubles during the first 25 years and then decreases to its original size over the succeeding 25 years. Thus, the population experiences (1) rapid growth, (2) gradual growth, (3) gradual decline, and (4) rapid decline during the 50-year period. Since the initial population of active employees is undermature, the simulated population experiences various undermature, approximately mature, and overmature statuses. Several statistics associated with the corresponding *mature* population for the underlying decrement assumptions are given at the bottom of Table 4–7 for comparison.

The average age of active employees begins at age 35, increases to 40.7 after 25 years, and to age 47.3 by the end of the 50-year period. The average for the mature population is age 41.3, a value reached by the simulated population during its 29th year. The average service period of employees begins at 5 years, increases to 9.7 years after 25 years, and to 16.6 years after 50 years. The average period of service for the mature population is 10.6 years, a value reached by the simulated population in its 30th year. These statistics show that even though the active population in its 30th year is not perfectly mature, its average age and service at that point are very nearly equal to that of the corresponding mature population. The speed with which the simulated population attains an approximately mature status is due, in part, to the assumption of multiple entry ages.

Table 4–7 also shows the total and average salaries of active employees, expressed as a percent of their respective values at the outset of the 50-year period. The ratio for total salaries increases to exactly double the ratio for average salaries after 25 years, a relationship consistent with the increase in the number of active employees. After 50 years, however, the total salary percentage is identical to the average salary percentage, since the population has returned to its initial size by this time. The rate of growth in average salary is approximately 5.9 percent during the first 25 years, and approximately 5.6 percent during the last 25 years.

TABLE 4–7

Population Statistics

Year	Number as Percent of Initial Size	Average Age	Average Service	Total Salary as Percentage of Initial	Average Salary as Percentage of Initial	Number as a Percent of Actives: Retired	Vested Terminated	Disabled	Surviving Spouses
0	100.0	35.0	5.0	100.0	100.0	0.0	0.0	0.0	0.0
1	107.8	34.8	4.9	114.6	106.3	1.7	2.4	0.0	0.1
2	115.4	35.0	5.0	130.9	113.5	2.0	5.0	0.1	0.2
3	122.6	35.3	5.2	148.1	120.8	2.3	7.8	0.1	0.3
4	129.4	35.6	5.3	166.2	128.5	2.7	10.9	0.1	0.4
5	136.0	35.9	5.5	185.5	136.4	3.1	14.2	0.2	0.5
6	142.2	36.2	5.7	205.8	144.7	3.5	17.8	0.2	0.7
7	148.2	36.5	5.9	227.3	153.4	4.0	21.6	0.3	0.8
8	153.8	36.8	6.1	250.0	162.5	4.5	25.7	0.3	1.0
9	159.0	37.0	6.3	273.7	172.1	5.0	30.0	0.4	1.2
10	164.0	37.3	6.6	298.9	182.2	5.6	34.4	0.4	1.4
11	168.6	37.6	6.8	325.1	192.8	6.2	39.1	0.5	1.6
12	173.0	37.8	7.0	352.9	204.0	6.9	43.8	0.6	1.8
13	177.0	38.1	7.2	381.8	215.7	7.6	48.7	0.7	2.1
14	180.6	38.3	7.4	412.0	228.1	8.3	53.7	0.8	2.3
15	184.0	38.6	7.6	443.6	241.1	9.1	58.9	0.9	2.6
16	187.0	38.8	7.9	476.5	254.8	9.9	64.1	1.0	2.9
17	189.8	39.0	8.1	510.9	269.2	10.8	69.3	1.1	3.2
18	192.2	39.3	8.3	546.5	284.4	11.7	74.6	1.3	3.5
19	194.2	39.5	8.5	583.3	300.3	12.7	79.9	1.4	3.8
20	196.0	39.7	8.7	621.5	317.1	13.7	85.3	1.6	4.1
21	197.4	39.9	8.9	660.9	334.8	14.7	90.6	1.7	4.5
22	198.6	40.1	9.1	701.6	353.3	15.9	95.9	1.9	4.8
23	199.4	40.3	9.3	743.3	372.8	17.1	101.2	2.1	5.2
24	199.8	40.5	9.5	785.9	393.3	18.3	106.5	2.3	5.6
25	200.0	40.7	9.7	829.8	414.9	19.5	111.7	2.5	5.9
26	199.8	40.9	9.9	874.5	437.7	20.8	116.8	2.7	6.3
27	199.4	41.0	10.1	920.1	461.4	22.2	121.8	2.9	6.7
28	198.6	41.2	10.2	966.3	486.6	23.6	126.8	3.1	7.1
29	197.4	41.3	10.4	1,012.8	513.1	24.9	131.6	3.2	7.4
30	196.0	41.5	10.6	1,060.3	541.0	26.3	136.3	3.4	7.8
31	194.2	41.7	10.8	1,107.7	570.4	27.6	140.9	3.6	8.2
32	192.2	41.8	10.9	1,155.5	601.2	29.1	145.3	3.8	8.5
33	189.8	42.0	11.1	1,202.9	633.8	30.5	149.5	4.0	8.9
34	187.0	42.1	11.3	1,249.7	668.3	31.8	153.6	4.2	9.2
35	184.0	42.3	11.5	1,296.5	704.6	33.1	157.5	4.4	9.6
36	180.6	42.5	11.7	1,342.1	743.1	34.4	161.1	4.5	9.9
37	177.0	42.7	11.8	1,386.8	783.5	35.6	164.6	4.7	10.2
38	173.0	42.9	12.0	1,429.6	826.4	36.8	167.8	4.8	10.5
39	168.6	43.1	12.2	1,470.1	871.9	38.0	170.8	5.0	10.8
40	164.0	43.3	12.5	1,509.0	920.1	39.0	173.6	5.1	11.0
41	159.0	43.5	12.7	1,544.6	971.5	40.0	176.1	5.2	11.3
42	153.8	43.7	13.0	1,577.2	1,025.5	41.0	178.3	5.3	11.5
43	148.2	44.0	13.2	1,605.4	1,083.3	41.8	180.2	5.4	11.7
44	142.2	44.3	13.5	1,628.6	1,145.3	42.6	181.8	5.5	11.9
45	136.0	44.7	13.9	1,647.6	1,211.4	43.3	183.1	5.6	12.0
46	129.4	45.0	14.3	1,659.8	1,282.7	43.9	184.1	5.6	12.2
47	122.6	45.5	14.7	1,666.0	1,358.9	44.4	184.7	5.7	12.3
48	115.4	46.0	15.2	1,663.8	1,441.8	44.8	185.0	5.7	12.4
49	107.8	46.6	15.8	1,652.2	1,532.6	45.1	185.0	5.8	12.5
50	100.0	47.3	16.6	1,632.1	1,632.1	45.3	184.6	5.8	12.5
Mature Population:		41.3	10.6			23.3	103.0	2.9	6.7

These values exceed 5 percent (i.e., 1 percent productivity and 4 percent inflation) because of the maturation of the population and its interaction with the merit salary scale.

The last four columns of Table 4–7 show the number of non-active plan members as a percentage of active employees. There are no nonactives at the outset of the simulation, by definition, but by the end of 25 years, retired employees total 19.5 percent, vested termination employees (in both a pre-and post-retirement status) total 111.7 percent, disabled employees total 2.5 percent, and surviving spouses total 5.9 percent. After 50 years, retired employees total 45.3 percent of active employees, a considerably overmature state in comparison to the 23.3 percent for the mature population (see the last row of Table 4–7). The corresponding number of vested terminated employees after 50 years is 184.6 percent, as compared to 103.0 percent for the mature population. The largest number of disabled employees and surviving spouses total only 5.8 and 12.5 percent, respectively, even for the extremely overmature population at the end of the 50-year period.