PENSION MATHEMATICS
with Numerical Illustrations

Second Edition

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Chapter 8

Ancillary Benefits

The mathematics associated with various ancillary benefits is presented in this chapter. The first section defines the one year *term cost* of each ancillary benefit and the present value of future benefits, both of which are relevant to defining the cost of ancillary benefits under various actuarial cost methods.

TERM COST CONCEPT

The one year term cost associated with a given ancillary benefit is the liability expected to be created during the upcoming year. For example, if the benefit under consideration is a lump sum death benefit, the term cost is equal to the cost of one year's term insurance in the amount of the death benefit. In the past, term cost methodology was often used because of its computational simplicity, even though the plan's retirement benefits were funded according to a particular actuarial cost method. With the advent of high speed computers, this methodology is seldom used for private plans; however, it is sometimes used with public pension plans. Term cost is used in this chapter as a precursor to determining the cost of ancillary benefits under various cost methods.

VESTED TERMINATION BENEFITS

Term Cost

The term cost (TC) of vested termination benefits for an employee age x is given by

$$^{v}(TC)_x = g_x^{(v)}\ B_x\ q_x^{(t)}\ {}_{r-x-1}p_{x+1}^{(m)}\ v^{r-x}\ \ddot{a}_r \tag{8.1}$$

where

$g_x^{(v)}$ = grading function equal to the proportion of accrued benefit vested at age x

B_x = accrued benefit at age x as defined by the plan benefit formula

$q_x^{(t)}$ = probability of terminating during age x

${}_{r-x-1}p_{x+1}^{(m)}$ = probability of living from age $x + 1$ to retirement.

This formulation shows that the term cost of vesting is the expected liability associated with the contingency that the employee may terminate vested during age x.[1] The term cost of vesting is zero until the first vesting age (denoted by z in this discussion). The grading function in (8.1) takes on the value of zero during this interval, resulting in a zero term cost. Additionally, the term cost is zero at and beyond the first early retirement qualification age, since the termination probability in (8.1) is zero at these ages. Generally, the cost pattern between these ages will be increasing; however, this need not be the case if the decrease in the attained age termination probability more than offsets the product of the survival probability and the interest discount function, both of which increase with age.

Present Value of Future Vested Termination Benefits

The PVFB for vested termination benefits can be expressed by taking the present value of the employee's future term cost of vesting:

$$^{v}(PVFB)_x = \sum_{k=x}^{r'-1} {}_{k-x}p_x^{(T)}\ v^{k-x}\ {}^{v}(TC)_k$$

$$= \left[\sum_{k=x}^{r'-1} g_k^{(v)}\ B_k\ {}_{k-x}p_x^{(T)}\ q_k^{(t)}\ {}_{r-k-1}p_{k+1}^{(m)}\right] v^{r-x}\ \ddot{a}_r. \tag{8.2}$$

[1]The grading function and accrued benefit function could be evaluated at age x + 1/2 to approximate a fractional year's credit. Similarly, the survival function could be evaluated at this fractional age as well. These refinements are not made in the mathematical notation in this chapter in order to simplify the equations.

This function increases until the first vesting age, after which it may or may not increase for a period of time (depending on the underlying actuarial assumptions) and then decreases to zero by r' (the first early retirement eligibility age).

DISABILITY BENEFITS

Term Cost

The term cost of disability for an employee age x can be expressed by

$$^{d}(TC)_x = g_x^{(d)}\ B_x\ q_x^{(d)}\ {}_w^d p_{x+1}^{(m)}\ v^{w+1}\ \ddot{a}_{x+w}^{d} \tag{8.3}$$

where

$g_x^{(d)}$ = grading function equal to the proportion of accrued benefit provided if disability occurs during age x

$q_x^{(d)}$ = probability of becoming disabled during age x

w = waiting period before disability benefits commence

${}_w^d p_{x+1}^{(m)}$ = probability that a *disabled* life age x lives w years

$\ddot{a}_{x+w}^{d}$ = life annuity based on disabled-life mortality.

This formulation assumes that the disability benefit is a function of the plan's accrued benefit. If this is not the case, the benefit function in (8.3) would be changed accordingly. The survival function and the annuity are both based on disabled-life mortality.

The term cost of disability will be zero prior to the first qualification age and will generally increase thereafter, although this need not be the case under some benefit formulas or actuarial assumptions.

Present Value of Future Disability Benefits

The present value of future disability benefits can be expressed by the following equation:

$$^{d}(PVFB)_x = \sum_{k=x}^{r-1} {}_{k-x}p_x^{(T)}\, v^{k-x}\, {}^{d}(TC)_k$$

$$= \left[\sum_{k=x}^{r-1} g_k^{(d)}\, B_k\, {}_{k-x}p_x^{(T)}\, q_k^{(d)}\, {}_{w}^{d}p_{k+1}^{(m)}\right] v^{k+w+1-x}\, \ddot{a}_{k+w+1}^{d}. \quad (8.4)$$

SURVIVING SPOUSE BENEFITS

Term Cost

Equation (8.5) represents the term cost of a surviving spouse benefit, where the benefit paid is a life annuity to the spouse commencing on the employee's death and equal to some fraction of the deceased employee's accrued benefit:

$$^{s}(TC)_x = M\, g_x^{(s)}\, B_x\, q_x^{(m)}\, v\, \ddot{a}_{x+u+1} \quad (8.5)$$

where

M = probability that the participant has a surviving spouse at death

$g_x^{(s)}$ = grading function equal to the proportion of accrued benefit provided to a surviving spouse if death occurs during age x

$q_x^{(m)}$ = probability of dying during age x

u = number of years (positive or negative) that, when added to the participant's age, yields an assumed age for the surviving spouse

$\ddot{a}_{x+u+1}$ = life annuity based on the spouse's age at the death of the participant.

The expected liability created by the possibility that the employee age x may die during the year has the same general form as the expected vested and disability liabilities. The coefficient M reflects the probability that the participant is married at the time of death, a probability frequently set in the 80 to 85 percent range. The grading function controls the portion of the accrued benefit payable to the surviving spouse at the participant's age x, and the spouse's annuity represents the cost of providing the benefit for the life of the spouse.

Present Value of Future Surviving Spouse Benefits

Taking the present value of future term costs associated with the surviving spouse benefit, we have the following expression for the present value of future benefits function:

$$^{s}(PVFB)_x = \sum_{k=x}^{r-1} {}_{k-x}p_x^{(T)} \, v^{k-x} \, {}^{s}(TC)_k$$

$$= M \left[\sum_{k=x}^{r-1} g_k^{(s)} \, B_k \, {}_{k-x}p_x^{(T)} \, q_k^{(m)} \right] v^{k+1-x} \, \ddot{a}_{k+u+1} \, . \quad (8.6)$$

Notice that (8.6) does not take into account the probability that the spouse survives to each potential future age for which the employee might die. Rather, the M coefficient is used to approximate the probability that the participant has a living spouse at all ages from age x to age $r-1$. It is possible to define M as the probability of having a spouse at age x only, and to use a survival function to account for the spouse's probability of living until the employee's age of death, but this is unduly complex given the relatively small cost of this ancillary benefit.

NUMERICAL ILLUSTRATION

Table 8–1 shows the term cost functions, expressed as a percentage of salary, and the present value of future benefit functions, expressed as a percentage of the retirement-based present value of future benefits function, for an age 30-entrant under the three ancillary benefits of the model plan.

ANCILLARY BENEFITS UNDER ACTUARIAL COST METHODS

The total cost of both ancillary and retirement benefits under various individual actuarial cost methods is considered in this section. The aggregate versions of each cost method are not presented, but the principles are the same.

Accrued Benefit Method

The normal cost under the accrued benefit method, consistent with the underlying theory of this method, is equal to the

TABLE 8–1

Ancillary Benefit Cost Functions:
Term Costs as a Percent of Salary and PVFB Functions as a Percent of $^{r}(PVFB)_x$

	Vested Termination Benefits		Disability Benefits		Surviving Spouse Benefits	
Age	Term Cost	$100 \cdot \frac{^{v}(PVFB)_x}{^{r}(PVFB)_x}$	Term Cost	$100 \cdot \frac{^{d}(PVFB)_x}{^{r}(PVFB)_x}$	Term Cost	$100 \cdot \frac{^{s}(PVFB)_x}{^{r}(PVFB)_x}$
30	0.00	19.76	0.00	16.84	0.00	10.42
31	0.00	19.76	0.00	16.84	0.00	10.42
32	0.00	19.76	0.00	16.84	0.00	10.42
33	0.00	19.76	0.00	16.84	0.00	10.42
34	0.00	19.76	0.00	16.84	0.00	10.42
35	0.33	19.76	0.00	16.84	0.01	10.42
36	0.40	19.28	0.00	16.84	0.01	10.41
37	0.46	18.75	0.00	16.84	0.01	10.40
38	0.53	18.17	0.00	16.84	0.02	10.38
39	0.60	17.55	0.00	16.84	0.02	10.36
40	0.69	16.88	0.12	16.84	0.03	10.33
41	0.77	16.17	0.14	16.72	0.04	10.30
42	0.87	15.41	0.18	16.59	0.05	10.26
43	0.98	14.61	0.23	16.42	0.07	10.21
44	1.10	13.75	0.28	16.22	0.09	10.15
45	1.25	12.84	0.33	15.99	0.12	10.08
46	1.40	11.87	0.38	15.74	0.16	9.99
47	1.58	10.83	0.44	15.46	0.20	9.87
48	1.78	9.73	0.53	15.15	0.26	9.73
49	2.00	8.56	0.62	14.80	0.34	9.55
50	2.26	7.32	0.71	14.41	0.43	9.34
51	2.53	6.00	0.81	14.00	0.54	9.09
52	2.84	4.60	0.94	13.55	0.68	8.79
53	3.16	3.13	1.07	13.07	0.85	8.44
54	3.51	1.60	1.20	12.55	1.05	8.03
55	0.00	0.00	1.37	12.00	1.26	7.55
56	0.00	0.00	1.51	11.41	1.41	7.02
57	0.00	0.00	1.72	10.79	1.57	6.44
58	0.00	0.00	1.99	10.10	1.74	5.81
59	0.00	0.00	2.39	9.33	1.96	5.14
60	0.00	0.00	2.98	8.45	2.20	4.41
61	0.00	0.00	3.83	7.39	2.47	3.63
62	0.00	0.00	5.01	6.09	2.75	2.80
63	0.00	0.00	6.59	4.48	3.07	1.91
64	0.00	0.00	8.65	2.47	3.42	0.97
65	0.00	0.00	0.00	0.00	0.00	0.00

present value of a deferred annuity of b_x (the formula benefit accrual), where the present value is based on the possibility that the employee may either terminate vested and be entitled to a deferred vested benefit, become disabled and receive a disability benefit, die and leave a surviving spouse with a benefit, or retire and receive a retirement benefit. The total normal cost for a participant age x can be represented by

$$^{AB}\,{}^{T}(NC)_x = \left[b_x \sum_{k=x}^{r-1} {}_{k-x}p_x^{(T)}\, v^{k-x} \left(q_k^{(t)}\, {}^{v}F_k + q_k^{(d)}\, {}^{d}F_k + q_k^{(m)}\, {}^{s}F_k \right) \right] + b_x\, {}^{r}F_r \quad (8.7)$$

where each F_k function represents the value of the benefit payable at each decrement:

$$\begin{aligned} {}^{v}F_k &= g_k^{(v)}\ {}_{r-k-1}p_{k+1}^{(m)}\ v^{r-k}\ \ddot{a}_r \\ {}^{d}F_k &= g_k^{(d)}\ {}_{w}^{d}p_{k+1}^{(m)}\ v^{w+1}\ \ddot{a}_{k+w+1}^{d} \\ {}^{s}F_k &= M\ g_k^{(s)}\ v^{1}\ \ddot{a}_{k+u+1} \\ {}^{r}F_r &= {}_{r-x}p_x^{(T)}\ v^{r-x}\ \ddot{a}_r. \end{aligned}$$

The actuarial liability under the accrued benefit method is found by substituting B_x for b_x in equation (8.7).

Benefit Prorate Methods

The normal cost under the constant dollar version of the benefit prorate method is given by

$$^{BD}\,{}^{T}(NC)_x = \left[\sum_{k=x}^{r-1} \frac{B_k}{(k-y)}\ {}_{k-x}p_x^{(T)}\, v^{k-x} \left(q_k^{(t)}\, {}^{v}F_k + q_k^{(d)}\, {}^{d}F_k + q_k^{(m)}\, {}^{s}F_k \right) \right] + \frac{B_r}{(r-y)}\, {}^{r}F_r. \quad (8.8)$$

The benefit accrual allocated to each age at which ancillary benefits are applicable is the projected benefit at that age divided by the years-of-service from entry age to that age. This factor is represented by the first term to the right of the summation sign in (8.8). Otherwise, the equation for the benefit prorate method is the same as that for the accrued benefit method.[2]

The actuarial liability is found by multiplying each of the prorated benefit components in (8.8) by the employee's year-of-

[2] An alternative methodology is to apply the current year's benefit accrual percentage to the employee's projected salary. If the benefit accrual is a constant percentage of salary for each year of service, this methodology is identical to the methodology presented here; however, if the benefit accrual is non-linear, then the alternative methodology would be the preferred approach.

service to date [i.e., $(x - y)$], which produces the appropriate accrued benefit under the constant dollar version of this method.[3]

The normal cost under the constant percent benefit prorate method is found by making the appropriate modifications to the prorated accrued benefit:

$$^{BP\,T}(NC)_x = \left[\sum_{k=x}^{r-1} \frac{B_k}{S_k}\, s_k\ {}_{k-x}p_x^{(T)}\, v^{k-x}\left(q_k^{(t)}\ {}^{v}F_k + q_k^{(d)}\ {}^{d}F_k + q_k^{(m)}\ {}^{s}F_k\right)\right] + \frac{B_r}{S_r}\, s_x\ {}^{r}F_r. \quad (8.9)$$

The actuarial liability is determined by substituting S_x for s_x in equation (8.9).

Cost Prorate Methods

The normal cost associated with ancillary benefits under either version of the cost prorate method is determined by dividing the age-y present value of future ancillary benefits by the appropriate employment-based life annuity. Thus, the normal cost under the constant dollar method for both retirement and ancillary benefits is as follows:

$$^{CD\,T}(NC)_x = \frac{^{v}(PVFB)_y + {}^{d}(PVFB)_y + {}^{s}(PVFB)_y + {}^{r}(PVFB)_y}{\ddot{a}^{T}_{y:\overline{r-y}|}}$$

$$= \frac{^{T}(PVFB)_y}{\ddot{a}^{T}_{y:\overline{r-y}|}}. \quad (8.10a)$$

The actuarial liability, determined prospectively, is given by

$$^{CD\,T}(AL)_x = {}^{T}(PVFB)_x - {}^{T}(NC)_x\ \ddot{a}^{T}_{x:\overline{r-x}|}. \quad (8.10b)$$

The percent of salary, K, used with the constant percent version is found by substituting $s_y\ {}^{s}\ddot{a}^{T}_{y:\overline{r-y}|}$ for $\ddot{a}^{T}_{x:\overline{r-x}|}$ in (8.10a), and the last term in equation (8.10b) would be $K s_x\ {}^{s}\ddot{a}^{T}_{x:\overline{r-x}|}$.

[3]There are other possible definitions of the benefit prorate method. For example, one could derive a constant dollar benefit accrual for each ancillary benefit, or one could derive an aggregate constant dollar benefit accrual for all benefits taken together. These equations were presented in the first edition of this book; however, since that time the above formulation has become the accepted approach.

If the actuarial liability associated with only vested benefits, for example, were found using equation (8.10b), it can be reasoned that its value would be negative at and beyond the first early retirement eligibility age. This is the case, since the present value of future vested benefits would be zero, while the present value of future vesting-related normal costs would continue to have a positive value up to age r. This same phenomenon may also occur with other ancillary benefits at older ages. There are three approaches to eliminating this effect if it is deemed inappropriate, especially for plans consisting of a substantial number of older employees where the negative actuarial liabilities associated with ancillary benefits may be non-trivial. One approach is to use the term cost method for ancillary benefits while maintaining the cost prorate method for retirement benefits and, possibly, one or more ancillary benefits. A second approach is to use the accrued benefit method for the ancillary benefits in question. A final approach is to develop normal costs for each ancillary benefit such that the present value of future benefits is amortized over the period for which such benefits are applicable. In the case of vested benefits, the employment-based annuity used in the denominator of (8.10a) would run from age y to age r'. Although a negative actuarial liability may still occur just prior to age r', it would not be substantial.

ANCILLARY BENEFITS FOR ALTERNATIVE LIABILITY MEASURES

The plan termination liability assumes that the plan is terminated on the date of the valuation; hence, ancillary benefits are not applicable. Ancillary benefits, however, are included in the plan continuation liability. The mathematics are identical to those of the actuarial liability under the accrued benefit cost method (i.e., equation (8.7) with B_x substituted for b_x).

COMPARISON OF ANCILLARY BENEFIT COSTS

Table 8–2 shows the cost of each ancillary benefit as a percent of retirement benefits under each normal cost method at various attained ages for an age-30 entrant. Table 8–3 provides these same data for the actuarial liabilities. Finally, Figure 8–1a shows normal costs with ancillary benefits included for the model pension plan, while Figure 8–1b shows the corresponding actuarial liabilities, each expressed as a percent of payroll.

TABLE 8–2

Ancillary Benefits Under Alternative Normal Costs as a Percent of Retirement Benefit Normal Cost

Age:	30	35	40	45	50	55	60
Vested Benefits							
^{AB}NC	185.40	174.64	96.66	51.76	20.89	0.00	0.00
^{BD}NC	47.41	45.88	32.58	21.32	10.25	0.00	0.00
^{BP}NC	115.25	109.89	67.92	39.44	16.98	0.00	0.00
^{CD}NC	19.76	19.76	19.76	19.76	19.76	19.76	19.76
^{CP}NC	19.76	19.76	19.76	19.76	19.76	19.76	19.76
Disability Benefits							
^{AB}NC	43.17	43.17	42.22	32.69	23.78	16.20	9.33
^{BD}NC	22.73	22.73	22.53	20.10	16.89	13.09	8.39
^{BP}NC	34.61	34.61	34.05	27.97	21.47	15.27	9.08
^{CD}NC	16.84	16.84	16.84	16.84	16.84	16.84	16.84
^{CP}NC	16.84	16.84	16.84	16.84	16.84	16.84	16.84
Survivor Benefits							
^{AB}NC	26.36	26.13	23.73	20.82	16.57	10.82	4.85
^{BD}NC	14.03	14.00	13.58	12.83	11.27	8.36	4.30
^{BP}NC	21.20	21.08	19.77	17.91	14.79	10.08	4.70
^{CD}NC	10.42	10.42	10.42	10.42	10.42	10.42	10.42
^{CP}NC	10.42	10.42	10.42	10.42	10.42	10.42	10.42

TABLE 8–3

Ancillary Benefits Under Alternative Actuarial Liabilities as a Percent of Retirement Benefit Actuarial Liability

Age:	30	35	40	45	50	55	60
Vested Benefits							
${}^{AB}AL$	0.00	185.40	102.30	55.38	23.53	0.00	0.00
${}^{BD}AL$	0.00	47.41	33.76	22.42	11.38	0.00	0.00
${}^{BP}AL$	0.00	115.25	71.24	41.92	19.03	0.00	0.00
${}^{CD}AL$	0.00	19.76	16.00	11.88	6.50	−0.60	−0.22
${}^{CP}AL$	0.00	19.76	14.38	9.67	4.19	−2.61	−1.08
Disability Benefits							
${}^{AB}AL$	0.00	43.17	43.17	33.65	24.62	16.89	10.10
${}^{BD}AL$	0.00	22.73	22.73	20.39	17.24	13.49	9.00
${}^{BP}AL$	0.00	34.61	34.61	28.62	22.12	15.86	9.80
${}^{CD}AL$	0.00	16.84	16.84	15.88	14.25	11.85	8.35
${}^{CP}AL$	0.00	16.84	16.84	15.61	13.80	11.36	7.99
Survivor Benefits							
${}^{AB}AL$	0.00	26.36	23.98	21.17	17.07	11.45	5.42
${}^{BD}AL$	0.00	14.03	13.63	12.93	11.49	8.73	4.75
${}^{BP}AL$	0.00	21.20	19.92	18.15	15.18	10.62	5.24
${}^{CD}AL$	0.00	10.42	10.31	10.03	9.27	7.47	4.35
${}^{CP}AL$	0.00	10.42	10.26	9.92	9.07	7.17	4.08

FIGURE 8–1a

Normal Costs With Ancillary Benefits Under Alternative Actuarial Cost Methods

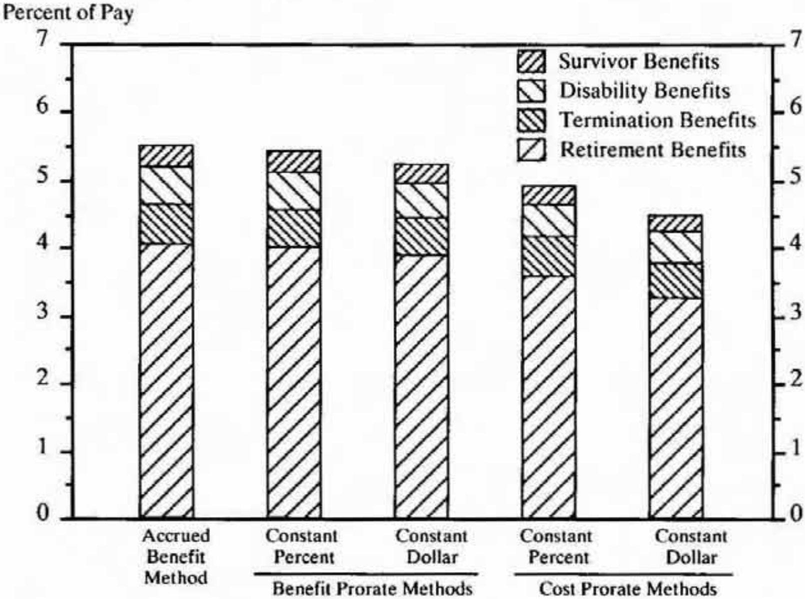

FIGURE 8–1b

Actuarial Liabilities With Ancillary Benefits Under Alternative Actuarial Cost Methods